Chapter 8 Applications of Newton's Second Law

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Those who are in love with practice without knowledge are like the sailor who gets into a ship without rudder or compass and who never can be certain whether he is going. Practice must always be founded on sound theory...[1]

Leonardo da Vinci

8.1 Force Laws

There are forces that don't change appreciably from one instant to another, which we refer to as constant in time, and forces that don't change appreciably from one point to another, which we refer to as constant in space. The gravitational force on an object near the surface of the earth is an example of a force that is constant in space.

There are forces that increase as you move away. When a mass is attached to one end of a spring and the spring is stretched a distance $|x|$, the spring force increases in strength proportional to the stretch.

There are forces that stay constant in magnitude but always point towards the center of a circle; for example when a ball is attached to a rope and spun in a horizontal circle with constant speed, the tension force acting on the ball is directed towards the center of the circle. This type of attractive central force is called a ***centripetal force***.

There are forces that spread out in space such that their influence becomes less with distance. Common examples are the gravitational and electrical forces. The gravitational force between two objects falls off as the inverse square of the distance separating the objects provided the objects are of a small dimension compared to the distance between them. More complicated arrangements of attracting and repelling things give rise to forces that fall off with other powers of r: constant, $1/r$, $1/r^2$, $1/r^3$, ...,.

How do we determine if there is any mathematical relationship, a ***force law*** that describes the relationship between the force and some measurable property of the objects involved?

8.1.1 Hooke's Law

We shall illustrate this procedure by considering the forces that compressed or stretched springs exert on objects. In order to stretch or compress a spring from its equilibrium length, a force must be exerted on the spring. Attach an object of mass m to one end of a spring and fix the other end of the spring to a wall (Figure 8.1). Let x_{eq} denote the equilibrium length of the spring (neither stretched or compressed). Assume that the

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[1] Notebooks of Leonardo da Vinci Complete, tr. Jean Paul Richter, 1888, Vol.1.

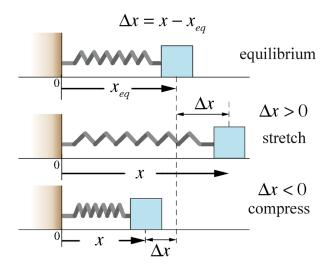contact surface is smooth and hence frictionless in order to consider only the effect of the spring force.

Figure 8.1 Spring attached to a wall and an object

Initially stretch the spring to a position x, corresponding to a displacement $\Delta x = x - x_{eq}$ (or compress the spring by $\Delta x < 0$), release the object, measure the acceleration, and then calculate the magnitude of the force of the spring acting on the object using the definition of force $|\vec{F}| = m|\vec{a}|$. Now repeat the experiment for a range of stretches (or compressions). Experiments will shown that for some range of lengths, $\Delta x_0 < \Delta x < \Delta x_1$, the magnitude of the measured force is proportional to the stretched length and is given by the formula

$$|\vec{F}| \propto |\Delta x| . \tag{8.1.1}$$

In addition, the direction of the acceleration is always towards the equilibrium position when the spring is neither stretched nor compressed. This type of force is called a **restoring force**. Let F_x denote the x-component of the spring force. Then

$$F_x = -k\Delta x = -k(x - x_{eq}) . \tag{8.1.2}$$

From the experimental data, the constant of proportionality, the **spring constant k**, can be determined. The spring constant has units $N \cdot m^{-1}$. The spring constant for each spring is determined experimentally by measuring the slope of the graph of the force *vs.* compression and extension stretch (Figure 8.2). Therefore for this one spring, the magnitude of the force is given by

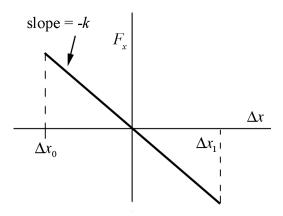$$|\vec{F}| = k|\Delta x| . \tag{8.1.3}$$

Figure 8.2 Plot of x-component of the spring force vs. compression and extension of spring

Now perform similar experiments on other springs. For a range of stretched lengths, each spring exhibits the same proportionality between force and stretched length, although the spring constant may differ for each spring.

It would be extremely impractical to experimentally determine whether this proportionality holds for all springs, and because a modest sampling of springs has confirmed the relation, we shall *infer* that all *ideal springs* will produce a restoring force, which is linearly proportional to the stretched (or compressed) length. This experimental relation regarding force and stretched (or compressed) lengths for a finite set of springs has now been inductively generalized into the above mathematical model for ideal springs, a model we refer to as a *force law*.

This inductive step, referred to as ***Newtonian induction,*** is the critical step that makes physics a predictive science. Suppose a spring, attached to an object of mass m, is stretched by an amount Δl. A prediction can be made, using the force law, that the magnitude of the force between the rubber band and the object is $\left|\vec{\mathbf{F}}\right| = k\left|\Delta x\right|$ without having to experimentally measure the acceleration. Now we can use Newton's second Law to predict the magnitude of the acceleration of the object

$$\left|\vec{\mathbf{a}}\right| = \frac{\left|\vec{\mathbf{F}}\right|}{m} = \frac{k\left|\Delta x\right|}{m}. \tag{8.1.4}$$

Now perform the experiment, and measure the acceleration within some error bounds. If the magnitude of the predicted acceleration disagrees with the measured result, then the model for the force law needs modification. The ability to adjust, correct or even reject models based on new experimental results enables a description of forces between objects to cover larger and larger experimental domains.

8.2 Fundamental Laws of Nature

Force laws are mathematical models of physical processes. They arise from observation and experimentation, and they have limited ranges of applicability. Does the linear force law for the spring hold for all springs? Each spring will most likely have a different range of linear behavior. So the model for stretching springs still lacks a universal character. As such, there should be some hesitation to generalize this observation to all springs unless some property of the spring, universal to all springs, is responsible for the force law.

Perhaps springs are made up of very small components, which when pulled apart tend to contract back together. This would suggest that there is some type of force that contracts spring molecules when they are pulled apart. What holds molecules together? Can we find some fundamental property of the interaction between atoms that will suffice to explain the macroscopic force law? This search for **fundamental forces** is a central task of physics.

In the case of springs, this could lead into an investigation of the composition and structural properties of the atoms that compose the steel in the spring. We would investigate the geometric properties of the lattice of atoms and determine whether there is some fundamental property of the atoms that create this lattice. Then we ask how stable is this lattice under deformations. This may lead to an investigation into the electron configurations associated with each atom and how they overlap to form bonds between atoms. These particles carry charges, which obey Coulomb's Law, but also the Laws of Quantum Mechanics. So in order to arrive at a satisfactory explanation of the elastic restoring properties of the spring, we need models that describe the fundamental physics that underline Hooke's Law.

8.2.1 Universal Law of Gravitation

At points significantly far away from the surface of the earth, the gravitational force is no longer constant with respect to the distance to the surface. **Newton's Universal Law of Gravitation** describes the gravitational force between two objects with masses, m_1 and m_2. This force points along the line connecting the objects, is attractive, and its magnitude is proportional to the inverse square of the distance, $r_{1,2}$, between the objects (Figure 8.3a). The force on object 2 due to the gravitational interaction between the two objects is given by

$$\vec{\mathbf{F}}^G_{1,2} = -G \frac{m_1 m_2}{r^2_{1,2}} \hat{\mathbf{r}}_{1,2} , \tag{8.2.1}$$

where $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$ is a vector directed from object 1 to object 2, $r_{1,2} = \left| \vec{\mathbf{r}}_{1,2} \right|$, and $\hat{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_{1,2} / \left| \vec{\mathbf{r}}_{1,2} \right|$ is a unit vector directed from object 1 to object 2 (Figure 8.3b). The constant of proportionality in SI units is $G = 6.67 \times 10^{-11} \, \text{N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$.
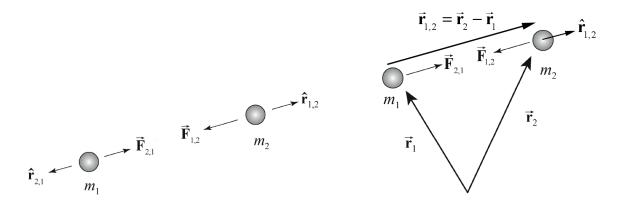
Figure 8.3 (a) Gravitational force between two objects.   **Figure 8.3 (b)** Coordinate system for the two-body problem.

8.2.2 Principle of Equivalence:

The Principle of Equivalence states that the mass that appears in the Universal Law of Gravity is identical to the inertial mass that is determined with respect to the standard kilogram. From this point on, the equivalence of inertial and gravitational mass will be assumed and the mass will be denoted by the symbol m.

8.2.3 Gravitational Force near the Surface of the Earth

Near the surface of the earth, the gravitational interaction between an object and the earth is mutually attractive and has a magnitude of

$$\left| \vec{\mathbf{F}}^{G}_{earth,object} \right| = m\,g \qquad (8.2.2)$$

where g is a positive constant.

The International Committee on Weights and Measures has adopted as a standard value for the acceleration of an object freely falling in a vacuum $g = 9.80665 \text{ m} \cdot \text{s}^{-2}$. The actual value of g varies as a function of elevation and latitude. If ϕ is the latitude and $h\,h$ the elevation in meters then the acceleration of gravity in SI units is

$$g = (9.80616 - 0.025928\cos(2\phi) + 0.000069\cos^2(2\phi) - 3.086 \times 10^{-4} h)\, \text{m} \cdot \text{s}^{-2}. \qquad (8.2.3)$$

This is known as Helmert's equation. The strength of the gravitational force on the standard kilogram at $42°$ latitude is $9.80345 \text{ N} \cdot \text{kg}^{-1}$, and the acceleration due to gravity at sea level is therefore $g = 9.80345 \text{ m} \cdot \text{s}^{-2}$ for all objects.   At the equator, $g = 9.78 \text{ m} \cdot \text{s}^{-2}$ (to three significant figures), and at the poles $g = 9.83 \text{ m} \cdot \text{s}^{-2}$. This difference is primarily due to the earth's rotation, which introduces an apparent

(fictitious) repulsive force that affects the determination of g as given in Equation (8.2.2) and also flattens the spherical shape earth (the distance from the center of the earth is larger at the equator than it is at the poles by about $26.5 \, \text{km}$). Both the magnitude and the direction of the gravitational force also show variations that depend on local features to an extent that's useful in prospecting for oil and navigating submerged nuclear submarines. Such variations in g can be measured with a sensitive spring balance. Local variations have been much studied over the past two decades in attempts to discover a proposed "fifth force" which would fall off faster than the gravitational force that falls off as the inverse square of the distance between the masses.

8.2.4 Electric Charge and Coulomb's Law

Matter has properties other than mass. As we have shown in the previous section, matter can also carry one of two types of observed *electric charge*, positive and negative. Like charges repel, and opposite charges attract each other. The unit of charge in the SI system of units is called the *coulomb* [C].

The smallest unit of "free" charge known in nature is the charge of an electron or proton, which has a magnitude of

$$e = 1.602 \times 10^{-19} \, \text{C} . \tag{8.2.4}$$

It has been shown experimentally that charge carried by ordinary objects is quantized in integral multiples of the magnitude of this free charge. The electron carries one unit of negative charge ($q_e = -e$) and the proton carries one unit of positive charge ($q_p = +e$).

In an isolated system, the charge stays constant; in a closed system, an amount of unbalanced charge can neither be created nor destroyed. Charge can only be transferred from one object to another.

Consider two objects with charges q_1 and q_2, separated by a distance $r_{1,2}$ in vacuum. By experimental observation, the two objects repel each other if they are both positively or negatively charged (Figure 8.4a). They attract each other if they are oppositely charged (Figure 8.4b). The force exerted on object 2 due to the interaction between 1 and 2 is given by Coulomb's Law,

$$\vec{\mathbf{F}}_{1,2}^{E} = k_e \frac{q_1 q_2}{r_{1,2}^2} \hat{\mathbf{r}}_{1,2} \tag{8.2.5}$$

where $\hat{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_{1,2} / \left| \vec{\mathbf{r}}_{1,2} \right|$ is a unit vector directed from object 1 to object 2, and in SI units, $k_e = 8.9875 \times 10^9 \, \text{N} \cdot \text{m}^2 \cdot \text{C}^{-2}$, as illustrated in the Figure 8.4a. This law was derived empirically by Charles Augustin de Coulomb in the late 18th century by the same methods as described in previous sections.
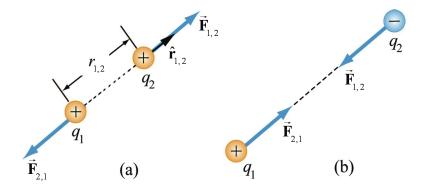
Figure 8.4 (a) and 8.4 (b) Coulomb interaction between two charges

Example 8.1 Coulomb's Law and the Universal Law of Gravitation

Show that Both Coulomb's Law and the Universal Law of Gravitation satisfy Newton's Third Law.

Solution: To see this, interchange 1 and 2 in the Universal Law of Gravitation to find the force on object 1 due to the interaction between the objects. The only quantity to change sign is the unit vector

$$\hat{\mathbf{r}}_{2,1} = -\hat{\mathbf{r}}_{1,2} . \tag{8.2.6}$$

Then

$$\vec{\mathbf{F}}_{2,1}^{G} = -G\frac{m_2\,m_1}{r_{2,1}^2}\hat{\mathbf{r}}_{2,1} = G\frac{m_1\,m_2}{r_{1,2}^2}\hat{\mathbf{r}}_{1,2} = -\vec{\mathbf{F}}_{1,2}^{G} . \tag{8.2.7}$$

Coulomb's Law also satisfies Newton's Third Law since the only quantity to change sign is the unit vector, just as in the case of the Universal Law of Gravitation.

8.3 Contact Forces

Pushing, lifting and pulling are ***contact forces*** that we experience in the everyday world. Rest your hand on a table; the atoms that form the molecules that make up the table and your hand are in contact with each other. If you press harder, the atoms are also pressed closer together. The electrons in the atoms begin to repel each other and your hand is pushed in the opposite direction by the table.

According to Newton's Third Law, the force of your hand on the table is equal in magnitude and opposite in direction to the force of the table on your hand. Clearly, if you push harder the force increases. Try it! If you push your hand straight down on the table, the table pushes back in a direction perpendicular (normal) to the surface. Slide your hand gently forward along the surface of the table. You barely feel the table pushing

upward, but you do feel the friction acting as a resistive force to the motion of your hand. This force acts tangential to the surface and opposite to the motion of your hand. Push downward and forward. Try to estimate the magnitude of the force acting on your hand.

The force of the table acting on your hand, $\vec{F}^{C} \equiv \vec{C}$, is called the **contact force.** This force has both a normal component to the surface, $\vec{N}$, called the **normal force,** and a tangential component to the surface, $\vec{f}$, called the **frictional force** (Figure 8.5).
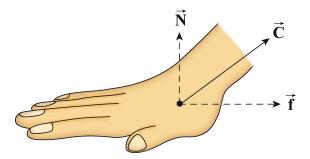
Figure 8.5 Normal and tangential components of the contact force

By the law of vector decomposition for forces,

$$\vec{C} \equiv \vec{N} + \vec{f}. \tag{8.3.1}$$

Any force can be decomposed into component vectors so the normal component, $\vec{N}$, and the tangential component, $\vec{f}$, are not independent forces but the vector components of the contact force perpendicular and parallel to the surface of contact.

In Figure 8.6, the forces acting on your hand are shown. These forces include the contact force, $\vec{C}$, of the table acting on your hand, the force of your forearm, $\vec{F}_{F,H} \equiv \vec{F}_{forearm}$, acting on your hand (which is drawn at an angle indicating that you are pushing down on your hand as well as forward), and the gravitational interaction, $\vec{F}_{E,H}^{G}$, between the earth and your hand.
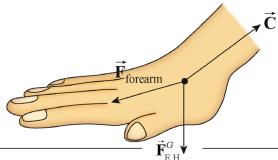
Figure 8.6 Forces on hand when moving towards the left

Is there a force law that mathematically describes this contact force? Since there are so many individual electrons interacting between the two surfaces, it is unlikely that we can add up all the individual forces. So we must content ourselves with a macroscopic model for the force law describing the contact force. One point to keep in mind is that the magnitudes of the two components of the contact force depend on how hard you push or pull your hand.

8.3.1 Free-body Force Diagram

When we try to describe forces acting on a collection of objects we must first take care to specifically define the collection of objects that we are interested in, which define our *system.* Often the system is a single isolated object but it can consist of multiple objects. Because force is a vector, the force acting on the system is a vector sum of the individual forces acting on the system

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots \tag{8.3.2}$$

A *free-body force diagram* is a representation of the sum of all the forces that act on a single system. We denote the system by a large circular dot, a "point". (Later on in the course we shall see that the "point" represents the center of mass of the system.) We represent each force that acts on the system by an arrow (indicating the direction of that force). We draw the arrow at the "point" representing the system. For example, the forces that regularly appear in free-body diagram are contact forces, tension, gravitation, friction, pressure forces, spring forces, electric and magnetic forces, which we shall introduce below. Sometimes we will draw the arrow representing the actual point in the system where the force is acting. When we do that, we will not represent the system by a "point" in the free-body diagram.

Suppose we choose a Cartesian coordinate system, then we can resolve the force into its component vectors

$$\vec{F} = F_x \, \hat{\mathbf{i}} + F_y \, \hat{\mathbf{j}} + F_z \, \hat{\mathbf{k}} \tag{8.3.3}$$

Each one of the component vectors is itself a vector sum of the individual component vectors from each contributing force. We can use the free-body force diagram to make these vector decompositions of the individual forces. For example, the x-component of the force is

$$F_x = F_{1,x} + F_{2,x} + \cdots . \tag{8.3.4}$$

Example 8.2 Tug-of-War

Two people, A and B, are competing in a tug-of-war (Figure 8.7). Person A is stronger, but neither person is moving because of friction. Draw separate free-body diagrams for each person (A and B), and for the rope. For each force on your free-body diagrams, describe the action-reaction force associated with it.

Figure 8.7 Tug-of-war

Solution: The forces acting on A are (Figure 8.8):

1. The force $\vec{\mathbf{F}}_{R,A}$ between A and the rope.

2. The gravitational force $\vec{\mathbf{F}}^G_{E,A} \equiv m_A \vec{\mathbf{g}}$ between A and the earth.

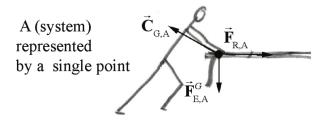3. The contact force $\vec{\mathbf{C}}_{G,A}$ between A and the ground.

Figure 8.8 Free-body force diagram on A

The forces acting on B are (Figure 8.9):

1. The force $\vec{\mathbf{F}}_{R,B}$ between B and the rope.

2. The gravitational force $\vec{\mathbf{F}}^G_{E,B} \equiv m_B \vec{\mathbf{g}}$ between B and the earth.

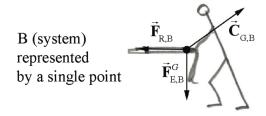3. The contact force $\vec{\mathbf{C}}_{G,B}$ between B and the ground.

Figure 8.9 Free-body force diagram on B

The forces on the rope are (Figure 8.10)

1. The force $\vec{\mathbf{F}}_{A,R}$ between the rope and A.

2. The force $\vec{\mathbf{F}}_{B,R}$ between the rope and B.

We neglect the gravitational force on the rope (massless rope assumption).

$$\vec{\mathbf{F}}_{A,R} \longleftarrow \text{Rope} \longrightarrow \vec{\mathbf{F}}_{B,R}$$

Figure 8.10 Free-body force diagram on rope

Because the rope is not moving, these forces sum to zero by Newton's Second Law but they are not Third Law pairs;
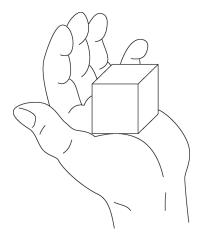
$$\vec{\mathbf{F}}_{A,R} + \vec{\mathbf{F}}_{B,R} = \vec{\mathbf{0}} .$$

The Newton's Third Law pairs are

1. The forces between A and the rope with $\vec{\mathbf{F}}_{A,R} = -\vec{\mathbf{F}}_{R,A}$.

2. The forces between B and the rope with $\vec{\mathbf{F}}_{B,R} = -\vec{\mathbf{F}}_{R,B}$.

3. The contact forces between A and the ground, $\vec{\mathbf{C}}_{A,G} = -\vec{\mathbf{C}}_{G,A}$. The contact force $\vec{\mathbf{C}}_{A,G}$ of A on the ground is not shown on the above force diagrams.

4. The contact forces between B and the ground, $\vec{\mathbf{C}}_{B,G} = -\vec{\mathbf{C}}_{G,B}$. The contact force $\vec{\mathbf{C}}_{B,G}$ of Team B on the ground is not shown on the above force diagrams.

5. The gravitational forces between the earth and A, $\vec{\mathbf{F}}_{E,A}^{G} = -\vec{\mathbf{F}}_{A,E}^{G}$. The gravitational force $\vec{\mathbf{F}}_{A,E}^{G}$ on the earth is not shown in the figures above.

6. The gravitational forces between the earth and B, $\vec{\mathbf{F}}_{E,B}^{G} = -\vec{\mathbf{F}}_{B,E}^{G}$. The gravitational force $\vec{\mathbf{F}}_{B,E}^{G}$ on the earth is not shown in the figures above.

8.3.3 Normal Component of the Contact Force and Weight

Hold an object in your hand. You can feel the "weight" of the object against your palm. But what exactly do we mean by "weight"? Consider the force diagram on the object in Figure 8.11. Let's choose the +-direction to point upward.
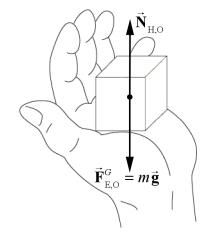
Figure 8.11 Object resting in hand

Figure 8.12 Force diagram on object

There are two forces acting on the object. One force is the gravitational force between the earth and the object, and is denoted by $\vec{\mathbf{F}}_{E,O}^{G} = m\vec{\mathbf{g}}$ where $\vec{\mathbf{g}}$, also known as the gravitational acceleration, is a vector that points downward and has magnitude $g = 9.8 \text{ m} \cdot \text{s}^{-2}$. The other force on the object is the contact force between your hand and the object. Because we are not pushing the block horizontally, this contact force on your hand points perpendicular to the surface, and hence has only a normal component, $\vec{\mathbf{N}}_{H,O}$. Let N denote the magnitude of the normal force. The force diagram on the object is shown in Figure 8.12. Because the object is at rest in your hand, the vertical acceleration is zero. Therefore Newton's Second Law states that

$$\vec{\mathbf{F}}_{E,O}^{G} + \vec{\mathbf{N}}_{H,O} = \vec{\mathbf{0}}. \qquad (8.3.5)$$

Choose the positive direction to be upwards, then in terms of vertical components we have that
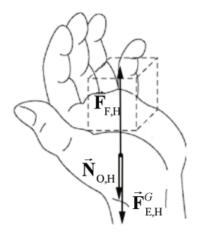
$$N - mg = 0, \qquad (8.3.6)$$

which can be solved for the magnitude of the normal force

$$N = mg. \qquad (8.3.7)$$

This result may give rise to a misconception that the normal force is always equal to the mass of the object times the magnitude of the gravitational acceleration at the surface of the earth. The normal force and the gravitational force are two completely different forces. In this particular example, the normal force is equal in magnitude to the gravitational force and directed in the opposite direction, which sounds like an example of the Third Law. But is it? No!

In order to see all the action-reaction pairs we must consider all the objects involved in the interaction. The extra object is your hand. The force diagram on your hand is shown in Figure 8.13.
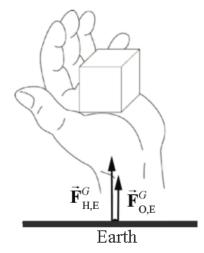

Figure 8.13 Free-body force diagram on hand

Figure 8.14 Gravitational forces on earth due to object and hand

The forces shown include the gravitational force between your hand and the earth, $\vec{\mathbf{F}}_{E,H}^{G}$ that points down, the normal force between the object and your hand, $\vec{\mathbf{N}}_{O,H}$, which also points down, and there is a force $\vec{\mathbf{F}}_{F,H}$ applied by your forearm to your hand that holds your hand up. There are also forces on the earth due to the gravitational interaction between the hand and object and earth. We show these forces in Figure 8.14: the gravitational force between the earth and your hand, $\vec{\mathbf{F}}_{H,E}^{G}$, and the gravitational force between the earth and the object, $\vec{\mathbf{F}}_{O,E}^{G}$.

There are three Third Law pairs. The first is associated with the normal force,

$$\vec{\mathbf{N}}_{O,H} = -\vec{\mathbf{N}}_{H,O} . \tag{8.3.8}$$

The second is the gravitational force between the mass and the earth,

$$\vec{\mathbf{F}}_{E,O}^{G} = -\vec{\mathbf{F}}_{O,E}^{G} . \tag{8.3.9}$$

The third is the gravitational force between your hand and the earth,

$$\vec{\mathbf{F}}_{E,H}^{G} = -\vec{\mathbf{F}}_{H,E}^{G} . \tag{8.3.10}$$

As we see, none of these three law pairs associates the "weight" of the block on the hand with the force of gravity between the block and the earth.

When we talk about the "weight" of an object, we often are referring to the effect the object has on a scale or on the feeling we have when we hold the object. These effects are actually effects of the normal force. We say that an object "feels lighter" if there is an additional force holding the object up. For example, you can rest an object in your hand, but use your other hand to apply a force upwards on the object to make it feel lighter in your supporting hand.

This leads us to the use of the word "weight," which is often used in place of the gravitational force that the earth, exerts on an object, and we will always refer to this force as the *gravitational force* instead of "weight." When you jump in the air, you feel "weightless" because there is no normal force acting on you, even though the earth is still exerting a gravitational force on you; clearly, when you jump, you do not turn gravity off! When astronauts are in orbit around the earth, televised images show the astronauts floating in the spacecraft cabin; the condition is described, rightly, as being "weightless." The gravitational force, while still present, has diminished slightly since their distance from the center of the earth has increased.

8.3.4 Static and Kinetic Friction

There are two distinguishing types of friction when surfaces are in contact with each other. The first type is when the two objects in contact are moving relative to each other; the friction in that case is called ***kinetic friction*** or ***sliding friction,*** and denoted by $\vec{\mathbf{f}}^k$.

Based on experimental measurements, the force of kinetic friction, $\vec{\mathbf{f}}^k$, between two surfaces, is independent of the relative speed of the surfaces, the area of contact, and only depends on the magnitude of the normal component of the contact force. The force law for the magnitude of the kinetic frictional force between the two surfaces can be modeled by

$$f_k = \mu_k N , \tag{8.3.11}$$

where μ_k is called the ***coefficient of kinetic friction***. The direction of kinetic friction on surface A due to the contact with a second surface B, $\vec{\mathbf{f}}^k_{B,A}$, is always opposed to the relative direction of motion of surface A with respect to the surface B.

The second type of friction is when the two surfaces are static relative to each other; the friction in that case is called ***static friction***, and denoted by $\vec{\mathbf{f}}^s$. Push your hand forward along a surface; as you increase your pushing force, the frictional force feels stronger and stronger. Try this! Your hand will at first stick until you push hard enough, then your hand slides forward. The magnitude of the static frictional force, f_s , depends on how hard you push.

If you rest your hand on a table without pushing horizontally, the static friction is zero. As you increase your push, the static friction increases until you push hard enough that your hand slips and starts to slide along the surface. Thus the magnitude of static friction can vary from zero to some maximum value, $(f_s)_{max}$, when the pushed object begins to slip,

$$0 \le f_s \le (f_s)_{max} . \tag{8.3.12}$$

Is there a mathematical model for the magnitude of the maximum value of static friction between two surfaces? Through experimentation, we find that this magnitude is, like kinetic friction, proportional to the magnitude of the normal force

$$(f_s)_{max} = \mu_s N . \tag{8.3.13}$$

Here the constant of proportionality is μ_s, the ***coefficient of static friction.*** This constant is slightly greater than the constant μ_k associated with kinetic friction, $\mu_s > \mu_k$. This small difference accounts for the slipping and catching of chalk on a blackboard, fingernails on glass, or a violin bow on a string.

The direction of static friction on an object is always opposed to the direction of the applied force (as long as the two surfaces are not accelerating). In Figure 8.15a, the static friction, $\vec{f}^s$, is shown opposing a pushing force, $\vec{F}_{B,O}^{push}$, acting on an object. In Figure 8.15b, when a pulling force, $\vec{F}_{B,O}^{pull}$, is acting on an object, static friction, $\vec{f}^s$, is now pointing the opposite direction from the pulling force.

pushing to the left
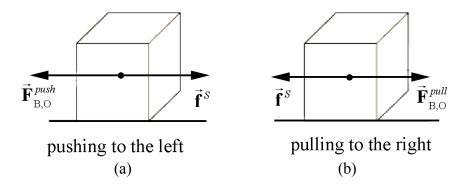(a)

pulling to the right
(b)

Figure 8.15 (a) and (b): Pushing and pulling forces and the direction of static friction.

Although the force law for the maximum magnitude of static friction resembles the force law for sliding friction, there are important differences:

1. The direction and magnitude of static friction on an object always depends on the direction and magnitude of the applied forces acting on the object, where the magnitude of kinetic friction for a sliding object is fixed.

2. The magnitude of static friction has a maximum possible value. If the magnitude of the applied force along the direction of the contact surface exceeds the magnitude of the maximum value of static friction, then the object will start to slip (and be subject to kinetic friction.) We call this the ***just slipping*** condition.

8.3.5 Modeling

One of the most central and yet most difficult tasks in analyzing a physical interaction is developing a physical model. A physical model for the interaction consists of a description of the forces acting on all the objects. The difficulty arises in deciding which forces to include. For example in describing almost all planetary motions, the Universal Law of Gravitation was the only force law that was needed. There were anomalies, for example the small shift in Mercury's orbit. These anomalies are interesting because they may lead to new physics. Einstein corrected Newton's Law of Gravitation by introducing General Relativity and one of the first successful predictions of the new theory was the perihelion precession of Mercury's orbit. On the other hand, the anomalies may simply be due to the complications introduced by forces that are well understood but complicated to model. When objects are in motion there is always some type of friction present. Air friction is often neglected because the mathematical models for air resistance are fairly complicated even though the force of air resistance substantially changes the motion. Static or kinetic friction between surfaces is sometimes ignored but not always. The mathematical description of the friction between surfaces has a simple expression so it can be included without making the description mathematically intractable. A good way to start thinking about the problem is to make a simple model, excluding complications that are small order effects. Then we can check the predictions of the model. Once we are satisfied that we are on the right track, we can include more complicated effects.

8.4 Tension in a Rope

How do we define "tension" in a rope? Let's return to our example of the very light rope ($m_R \simeq 0$) that is attached to a block B of mass m_B on one end, and pulled by an applied force by another object A, $\vec{F}_{A,R}$, from the other end (Figure 8.16). Let's assume that the block and rope are both at rest.
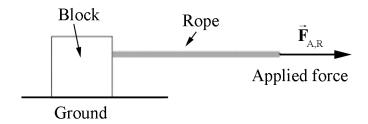

Figure 8.16 Forces acting on block and rope

Let's choose an xy-coordinate system with the $\hat{\mathbf{j}}$-unit vector pointing upward in the normal direction to the surface, and the $\hat{\mathbf{i}}$-unit vector pointing in the direction of the motion of the block. The force diagrams for the rope and block are shown in Figure 8.17.
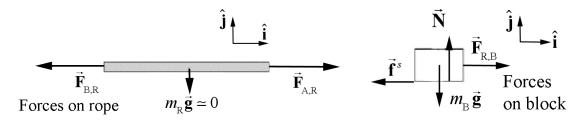

Figure 8.17 Force diagram for rope and block

The forces on the rope and the block must each sum to zero. Because the rope is not accelerating, Newton's Second Law becomes:

$$F_{A,R} - F_{B,R} = 0.\tag{8.4.1}$$

The static equilibrium conditions for the block are:

in the $+\hat{\mathbf{i}}$-direction:

$$F_{R,B} - f_s = 0,\tag{8.4.2}$$

in the $+\hat{\mathbf{j}}$-direction:

$$N - m_B g = 0.\tag{8.4.3}$$

We now apply Newton's Third Law, the action-reaction law,

$$\vec{\mathbf{F}}_{B,R} = -\vec{\mathbf{F}}_{R,B},\tag{8.4.4}$$

which becomes, in terms of our magnitudes,

$$F_{B,R} = F_{R,B}.\tag{8.4.5}$$

Note there is no minus sign in Equation (8.4.5) because these are magnitudes. The sign is built into Eqs. (8.4.1) and (8.4.2). Our static equilibrium conditions now become

$$F_{A,R} = F_{B,R} = F_{R,B} = f_s.\tag{8.4.6}$$

Thus the applied pulling force is transmitted through the rope to the object since it has the same magnitude as the force of the rope on the object.

$$F_{A,R} = F_{R,B}.\tag{8.4.7}$$

In addition we see that the applied force is equal to the static friction,

$$F_{A,R} = f_s. \tag{8.4.8}$$

8.4.1 Static Tension in a Rope

We have seen that in static equilibrium the pulling force by another object A, transmits through the rope. Suppose we make an imaginary slice of the rope at a distance x from the end that the object is attached to the object (Figure 8.18).
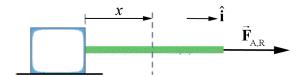

Figure 8.18 Imaginary slice through the rope

The rope is now divided into two sections, labeled left and right. Aside from the Third Law pair of forces between the object and the rope, there is now a Third Law pair of forces between the left section of the rope and the right section of the rope. We denote this force acting on the left section by $\vec{F}_{R,L}(x)$. The force on the right section due to the left section is denoted by $\vec{F}_{L,R}(x)$. Newton's Third Law requires that each force in this action-reaction pair is equal in magnitude and opposite in direction.

$$\vec{F}_{R,L}(x) = -\vec{F}_{L,R}(x) \tag{8.4.9}$$

The force diagram for the left and right sections are shown in Figure 8.19 where $\vec{F}_{B,L}$ is the force on the left-segment the rope due to the block-rope interaction.
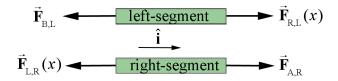

Figure 8.19 Force diagram for the left and right sections of rope

*The **tension** $T(x)$ in a rope at a distance x from one end of the rope is the magnitude of the action -reaction pair of forces acting at the point x,*

$$T(x) = \left| \vec{F}_{R,L}(x) \right| = \left| \vec{F}_{L,R}(x) \right|. \tag{8.4.10}$$

Special case: For a rope of negligible mass in static equilibrium, the tension is uniform and is equal to the applied force,

$$T = F_{A,R} . \tag{8.4.11}$$

Example 8.3 Tension in a Massive Rope

Suppose a uniform rope of mass m and length d is attached to an object of mass m_B lying on a table. The rope is pulled from the side opposite the block with an applied force of magnitude $\left| \vec{F}_{A,R} \right| = F_{A,R}$. The coefficient of kinetic friction between the block and the surface is μ_k. Find the tension in the rope as a function of distance from the block.

Solution: The key point to realize is that the rope is now massive. Suppose we make an imaginary slice of the rope at a distance x from the end that the object as we did in Figure 8.18. The mass of the right slice is given by

$$m_{right} = \frac{m}{d}(d - x) . \tag{8.4.12}$$

The mass of the left slice is

$$m_{left} = \frac{m}{d}x . \tag{8.4.13}$$

We now apply Newton's Second Law to the right slice of the rope

$$F_{A,R} - T(x) = \frac{m}{d}(d - x)a_{R,x} , \tag{8.4.14}$$

where $T(x)$ is the tension in the rope at a distance x from the object and $a_{R,x}$ is the x-component of the acceleration of the right piece of the rope. We also apply Newton's Second Law to the left slice of the rope

$$T(x) - F_{B,L} = \frac{m}{d}xa_{L,x} , \tag{8.4.15}$$

where $T(x)$ is the tension in the rope at a distance x from the object, $F_{B,L}$ is the magnitude of the force on the left-segment of the rope due to the rope-block interaction, and $a_{L,x}$ is the x-component of the acceleration of the left piece of the rope. The force diagram is shown in Figure 8.20.
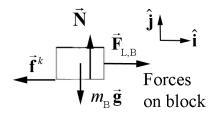
Figure 8.20 Force diagram on sliding block

Newton's Second Law on the block is now:

in the $+\hat{\mathbf{i}}$-direction:

$$F_{L,B} - f_k = m_B a_{B,x},$$ (8.4.16)

in the $+\hat{\mathbf{j}}$ -direction:

$$N - m_B g = 0.$$ (8.4.17)

Eq. (8.4.17) implies that $N = m_B g$ and so the kinetic friction force acting on the block is

$$f_k = \mu_k N = \mu_k m_B g.$$ (8.4.18)

We now substitute Eq. (8.4.18) into Eq. (8.4.16), which becomes

$$F_{L,B} = \mu_k m_B g + m_B a_{B,x},$$ (8.4.19)

Newton's Third Law for the block-rope interaction is given by

$$F_{L,B} = F_{B,L}.$$ (8.4.20)

We substitute Eq. (8.4.20) into Eq. (8.4.19) and then into Eq. (8.4.15) yielding

$$T(x) - (\mu_k m_B g + m_B a_{B,x}) = \frac{m}{d} x a_{L,x}.$$ (8.4.21)

Because the rope and block move together, the accelerations are equal which we denote by the symbol a

$$a \equiv a_{B,x} = a_{L,x}.$$ (8.4.22)

Then Eq. (8.4.21) becomes

$$T(x) = \mu_k m_B g + (m_B + \frac{m}{d} x)a.$$ (8.4.23)

Check our result: We could have used Eq. (8.4.14) to find the tension

$$T(x) = F_{A,R} - \frac{m}{d}(d-x)a . \qquad (8.4.24)$$

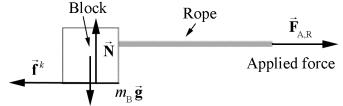The force diagram on the system consisting of the rope and block is shown in Figure 8.21.

Figure 8.21 Force diagram on block-rope system

Therefore Newton's Second Law becomes

$$F_{A,R} - f_k = (m + m_B)a . \qquad (8.4.25)$$

Therefore substitute Eq. (8.4.25) into Eq. (8.4.24) yielding

$$T(x) = f_k + (m + m_B)a - \frac{m}{d}(d-x)a = f_k + (m_B + \frac{m}{d}x)a . \qquad (8.4.26)$$

in agreement with Eq. (8.4.23). We expect this result because the tension is accelerating both the left slice and the block and is opposed by the frictional force.

Example 8.4 Tension in a Suspended Rope

Suppose a uniform rope of mass M and length L is suspended from a ceiling (Figure 8.22). The magnitude of the acceleration due to gravity is g. (a) Find the tension in the rope at the upper end where the rope is fixed to the ceiling. (b) Find the tension in the rope as a function of the distance from the ceiling. (c) Find an equation for the rate of change of the tension with respect to distance from the ceiling in terms of M, L, and g. Be sure you show any free-body force diagrams or sketches that you plan to use.
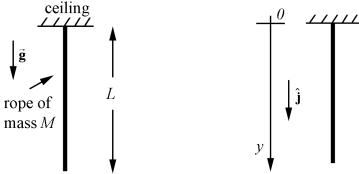

Figure 8.22 Rope suspended from ceiling   **Figure 8.23** Coordinate system for suspended rope

Solution: (a) We begin by choosing a coordinate system with the origin at the ceiling and the positive y-direction pointing downward (Figure 8.23). In order to find the tension at the upper end of the rope, we draw a free-body diagram of the entire rope. The forces acting on the rope are the force at $y = 0$ holding the rope up, (we refer to that force as $T(y = 0)$, the tension at the upper end) and the gravitational force on the entire rope $Mg\,\hat{\mathbf{j}}$. The free-body force diagram is shown in Figure 8.24.
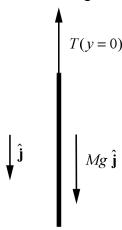

Figure 8.24 Force diagram on rope

We now apply Newton's Second Law noting that the acceleration is zero

$$Mg - T(y = 0) = 0.$$

Thus we can solve for $T(y = 0)$, the tension at the upper end,

$$T(y = 0) = Mg.$$

(b) Recall that the tension at a point is the magnitude of the action-reaction pair of forces acting at that point. So we make an imaginary cut in the rope a distance y from the ceiling separating the rope into an upper and lower piece (Figure 8.25). We choose the upper piece as our system with mass $m = (M / L)y$. The forces acting on the upper rope are the gravitational force $mg\,\hat{\mathbf{j}} = (M / L)yg\hat{\mathbf{j}}$, the force at $y = 0$ holding the rope up, (we refer to that force as $T(y = 0)$, the tension at the upper end), and the tension at the point y, $T(y)$ that is pulling the upper piece down. The free-body force diagram is shown in Figure 8.26.
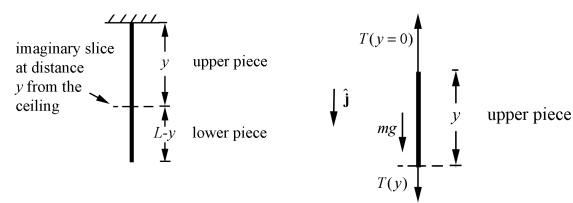
Figure 8.25 Imaginary slice separates rope into two pieces

Figure 8.26 Free-body force diagram on upper piece of rope

We now apply Newton's Second Law noting that the acceleration is zero

$$mg + T(y) - T(y = 0) = 0.$$

Thus we can solve for $T(y)$, the tension at a distance y from the ceiling,

$$T(y) = T(y = 0) - mg.$$

Using our results for the mass of the upper piece and the tension at the upper end we have that

$$T(y) = Mg(1 - y / L) \qquad (8.4.27)$$

As a check, we note that when $y = L$, the tension $T(y = L) = 0$ which is what we expect because there is no force acting at lower end of the rope.
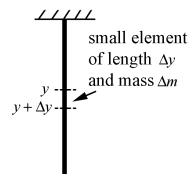
(c) We can differentiate Eq. (8.4.27) with respect to y and find that

$$\frac{dT}{dy}(y) = -(M / L)g. \qquad (8.4.28)$$

So the rate that the tension is changing is constant.

8.4.2 Continuous Systems and Newton's Second Law as a Differential Equations

We can determine the tension at a distance y from the ceiling in Example 8.4, by an alternative method, a technique that will generalize to many types of "continuous systems". We consider as our system an imaginary segment of the rope between the points y and $y + \Delta y$. This small element has length Δy and mass $\Delta m = (M / L)\Delta y$.
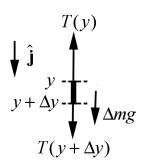
Figure 8.27 Rope with small element identified

Figure 8.28 Infinitesimal slice of rope

The rope has now been divided into three pieces, an upper piece, the small element, and a lower piece (Figure 8.27). The forces acting on it are the tension $T(y)$ at y directed upward, (the force of the upper piece holding the element up), the tension $T(y + \Delta y)$ at $y + \Delta y$ directed downward (the force of the lower piece pulling the element down), and the gravitational force $\Delta mg\, \hat{\mathbf{j}} = (M / L)\Delta yg\hat{\mathbf{j}}$. The free-body force diagram is shown in Figure 8.28.

We now apply Newton's Second Law to the small element noting that the acceleration is zero

$$\Delta mg + T(y + \Delta y) - T(y) = 0 .$$

We now solve for the difference in the tension

$$T(y + \Delta y) - T(y) = -\Delta mg .$$

We now substitute our result for the mass of the element $\Delta m = (M / L)\Delta y$, and find that

$$T(y + \Delta y) - T(y) = -(M / L)\Delta yg .$$

We now divide through by Δy yielding

$$\frac{T(y + \Delta y) - T(y)}{\Delta y} = -(M / L)g .$$

Now here comes the crucial step, the limiting argument. We consider the limit in which the length of the small element goes to zero, $\Delta y \to 0$.

$$\lim_{\Delta y \to 0} \frac{T(y + \Delta y) - T(y)}{\Delta y} = -(M / L)g .$$

Recall that the left hand side is the definition of the derivative of the tension with respect to y, and so we arrive at

$$\frac{dT}{dy}(y) = -(M/L)g$$

in agreement with Eq. (8.4.28).

We can solve this differential equation by a technique called **separation of variables**. We rewrite the equation as

$$dT = -(M/L)g\,dy$$

and integrate both sides. Our integral will be a definite integral, the limits of the right hand side are from $y = 0$ to y, and the corresponding limits on the left hand side are from $T(y = 0)$ to $T(y)$:

$$\int_{T(y=0)}^{T(y)} dT = -(M/L)g\int_{y'=0}^{y'=y} dy'.$$

Notice that we have introduced a "dummy" integration variable y' to distinguish the integration variable from the endpoint of the integral y, the point that we are trying to find the tension, $T(y)$. We now integrate and find that

$$T(y) - T(y = 0) = -(M/L)gy.$$

We use our earlier result that $T(y = 0) = Mg$ and find that

$$T(y) = Mg(1 - y/L).$$

in agreement with our earlier result.

8.5 Frictional Force as a Linear Function of Velocity

In many physical situations the force on an object will be modeled as depending on the object's velocity. We have already seen static and kinetic friction between surfaces modeled as being independent of the surfaces' relative velocity. Common experience (swimming, throwing a Frisbee) tells us that the frictional force between an object and a fluid can be a complicated function of velocity. Indeed, these complicated relations are an important part of such topics as aircraft design.

A reasonable model for the frictional force on an object m moving at low speeds through a viscous medium is

$$\vec{\mathbf{F}}_{\text{friction}} = -\gamma\, m\, \vec{\mathbf{v}} \tag{8.5.1}$$

where γ is a constant that depends on the properties (density, viscosity) of the medium and the size and shape of the object. Note that γ has dimensions of inverse time,

$$\dim[\gamma] = \frac{\dim\left[\text{Force}\right]}{\dim[\text{mass}] \cdot \dim\left[\text{velocity}\right]} = \frac{M \cdot L \cdot T^{-2}}{M \cdot L \cdot T^{-1}} = T^{-1}. \qquad (8.5.2)$$

The minus sign in Equation (8.5.1) indicates that the frictional force is directed against the object's velocity (relative to the fluid).

In a situation where $\vec{F}_{\text{friction}}$ is the net force, Newton's Second Law becomes

$$-\gamma\, m\, \vec{v} = m\vec{a} \qquad (8.5.3)$$

and so the acceleration is

$$\vec{a} = -\gamma\, \vec{v}. \qquad (8.5.4)$$

The acceleration has no component perpendicular to the velocity, and in the absence of other forces the object will move in a straight line, but with varying speed. Denote the direction of this motion as the x-direction, so that Equation (8.5.4) becomes

$$a_x = \frac{d v_x}{dt} = -\gamma\, v_x. \qquad (8.5.5)$$

Equation (8.5.5) is now a differential equation. For our purposes, we'll create an initial-condition problem by specifying that the initial x-component of velocity is $v(t=0) = v_{x0}$. The differential equation in (8.5.5) is also separable, in that the equation may be rewritten as

$$\frac{d v_x}{v_x} = -\gamma dt. \qquad (8.5.6)$$

and each side can be separately integrated. The integration in this case is simple, leading to

$$\int_{v_{x0}}^{v_x(t)} \frac{d v_x}{v_x} = -\gamma \int_0^t dt. \qquad (8.5.7)$$

The left hand side is

$$\text{LHS} = \ln(v_x)\Big|_{v_{x0}}^{v_x(t)} = \ln(v_x(t)) - \ln(v_{x0}) = \ln(v_x(t)/v_{x0}), \qquad (8.5.8)$$

and the right hand side is

$$\text{RHS} = -\gamma t. \qquad (8.5.9)$$

Equating the two sides yields

$$\ln(v_x(t)/v_{x0}) = -\gamma t. \tag{8.5.10}$$

Exponentiate each side of the above equation yields

$$v_x(t)/v_{x0} = e^{-\gamma t}. \tag{8.5.11}$$

Thus the x-component of the velocity as a function of time is given by

$$v_x(t) = v_{x0}e^{-\gamma t} \tag{8.5.12}$$

A plot of v_x vs. t is shown in Figure 8.29 with initial conditions $v_{x0} = 10 \text{ m} \cdot \text{s}^{-1}$ and $\gamma = 0.5 \text{ s}^{-1}$.
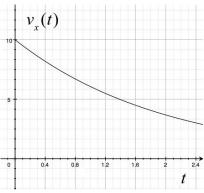

Figure 8.29 Plot of v_x vs. t for damping force $\vec{F}_{\text{friction}} = -\gamma m \vec{v}$

Example 8.5 Newtonian Damping

An object moving along the x-axis with an initial x-component of the velocity $v_x(t = 0) = v_{x0}$ experiences a retarding frictional force whose magnitude is proportional to the square of the speed (a case known as ***Newtonian Damping***),

$$\left| \vec{F}_{\text{friction}} \right| = \gamma m v^2 \tag{8.5.13}$$

Show that the x-component of the velocity of the object as a function of time is given by

$$v_x(t) = v_{x0} \frac{1}{1 + t/\tau}, \tag{8.5.14}$$

and find the constant τ.

Solution: Newton's Second Law can be written as

$$-\gamma v_x^2 = \frac{d v_x}{dt}.$$

(8.5.15)

Differentiating our possible solution yields

$$\frac{d v_x}{dt} = -v_{x0} \frac{1}{\tau} \frac{1}{\left(1 + t/\tau\right)^2} = -\frac{1}{v_{x0}\,\tau} v_x^2$$

(8.5.16)

Substituting into the Second Law yields

$$-\gamma v_x^2 = -\frac{1}{v_{x0}\,\tau} v_x^2.$$

(8.5.17)

Thus our function is a solution providing that

$$\tau = \frac{1}{v_{x0}\gamma}.$$

(8.5.18)

A plot of v_x vs. t is shown in Figure 8.30 with initial conditions $v_{x0} = 10\,\text{m}\cdot\text{s}^{-1}$ and $\gamma = 0.5\,\text{s}^{-1}$.
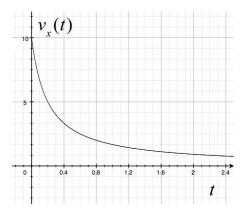

Figure 8.30 Plot of v_x vs. t for damping force $\left|\vec{\mathbf{F}}_{\text{friction}}\right| = \gamma\,m v^2$

Figure 8.31 shows the two x-component of the velocity functions Eq. (8.5.12) and Eq. (8.5.14) plotted on the same graph.
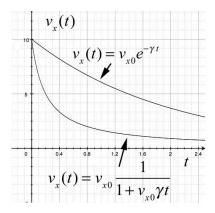
$$v_x(t)$$

$$v_x(t) = v_{x0}e^{-\gamma t}$$

$$v_x(t) = v_{x0}\frac{1}{1+v_{x0}\gamma t}$$

Figure 8.31 Plot of two different x-component of the velocity functions

8.6 Worked Examples

Example 8.6 Staircase

An object of given mass m starts with a given velocity v_0 and slides an unknown distance s along a floor and then off the top of a staircase (Figure 8.32). The goal of this problem is to find the distance s. The coefficient of kinetic friction between the object and the floor is given by μ_k. The object strikes at the far end of the third stair. Each stair has a given rise of h and a given run of d. Neglect air resistance and use g for the gravitational constant. (a) Briefly describe how you intend to model the motion of the object. What are the given quantities in this problem? (b) What is the distance s that the object slides along the floor? Express your answer in terms of the given quantities only.
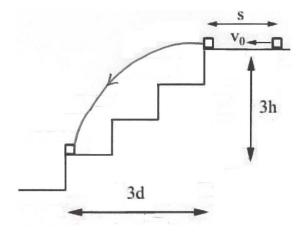

Figure 8.32 Object falling down a staircase

Solution: a) There are two distinct stages to the object's motion, the initial horizontal motion (the floor must be assumed horizontal) and the motion in free fall. The given

final position of the object, at the far end of the third stair, will determine the horizontal component of the velocity at the instant the object left the top of the stair. This in turn determined the time the object decelerated, and the deceleration while on the floor determined the distance traveled on the floor. The given quantities are m, v_0, μ_k, g, h and d.

b) From the top of the stair to the far end of the third stair, the object is in free fall. Take the positive $\hat{\mathbf{i}}$-direction to be horizontal, directed to the left in the figure, take the positive $\hat{\mathbf{j}}$-direction to be vertical (up) and take the origin at the top of the stair, where the object first goes into free fall. The components of acceleration are $a_x = 0$, $a_y = -g$, the initial x-component of velocity will be denoted $v_{x,0}$, the initial y-component of velocity is $v_{y,0} = 0$, the initial x-position is $x_0 = 0$ and the initial y-position is $y_0 = 0$. The equations describing the object's motion as a function of time t are then

$$x(t) = x_0 + v_{x,0}\,t + \frac{1}{2}a_x\,t^2 = v_{x,0}\,t \tag{8.6.1}$$

$$y(t) = y_0 + v_{y,0}\,t + \frac{1}{2}a_y\,t^2 = -\frac{1}{2}gt^2. \tag{8.6.2}$$

It's crucial to see that in this notation and that given in the problem, $v_{x,0} < v_0$. In the above equations, Eq. (8.6.1) may be solved for t to give

$$t = \frac{x(t)}{v_{x,0}}. \tag{8.6.3}$$

Substituting Eq. (8.6.3) into Eq. (8.6.2) and eliminating the variable t,

$$y(t) = -\frac{1}{2}g\frac{x^2(t)}{v_{x,0}^{\,2}}. \tag{8.6.4}$$

Eq. (8.6.4) can now be solved for the square of the horizontal component of the velocity,

$$v_{x,0}^{\,2} = -\frac{1}{2}g\frac{x^2(t)}{y(t)}. \tag{8.6.5}$$

At the far end of the third stair, $x = 3d$ and $y = -3h$; substitution into Eq. (8.6.5) gives

$$v_{x,0}^{\,2} = \frac{3gd^2}{2h}. \tag{8.6.6}$$

For the horizontal motion, use the same coordinates with the origin at the edge of the landing. The forces on the object are gravity $m\vec{g} = -mg\,\hat{j}$, the normal force $\vec{N} = N\,\hat{j}$ and the kinetic frictional force $\vec{f}_k = -f_k\,\hat{i}$. The components of the vectors in Newton's Second Law, $\vec{F} = m\vec{a}$, are

$$-f_k = m\,a_x$$
$$N - m\,g = m\,a_y. \qquad (8.6.7)$$

The object does not move in the y-direction; $a_y = 0$ and thus from the second expression in (8.6.7), $N = m\,g$. The magnitude of the frictional force is then $f_k = \mu_k N = \mu_k mg$, and the first expression in (8.6.7) gives the x-component of acceleration as $a_x = -\mu_k g$. Since the acceleration is constant the x-component of the velocity is given by

$$v_x(t) = v_0 + a_x\,t, \qquad (8.6.8)$$

where v_0 is the x-component of the velocity of the object when it just started sliding (this is a different from the x-component of the velocity when it just leaves the top landing, a quantity we denoted earlier by $v_{x,0}$). The displacement is given by

$$x(t) - x_0 = v_0\,t + \frac{1}{2}a_x\,t^2. \qquad (8.6.9)$$

Denote the time the block is decelerating as t_1. The initial conditions needed to solve the problem are the given v_0 (assuming we've "reset" our clocks appropriately), $v_x(t_1) = v_{x,0}$ as found above, $x_0 = -s$ and $x(t_1) = 0$. (Note that in the conditions for the initial and final position, we've kept the origin at the top of the stair. This is not necessary, but it works. The important matter is that the x-coordinate increases from right to left.) During this time, the block's x-component of velocity decreases from v_0 to $v_x(t_1) = v_{x,0}$ with constant acceleration $a_x = -\mu_k g$. Using the initial and final conditions, and the value of the acceleration, Eq. (8.6.9) becomes

$$s = v_0\,t_1 - \frac{1}{2}\mu_k g\,t_1^2 \qquad (8.6.10)$$

and we can solve Eq. (8.6.8) for the time the block reaches the edge of the landing,

$$t_1 = \frac{v_{x,0} - v_0}{-\mu_k g} = \frac{v_0 - v_{x,0}}{\mu_k g}. \qquad (8.6.11)$$

Substituting Eq. (8.6.11) into Eq. (8.6.10) yields

$$s = v_0 \left(\frac{v_0 - v_{x,0}}{\mu_k g} \right) - \frac{1}{2} \mu_k g \left(\frac{v_0 - v_{x,0}}{\mu_k g} \right)^2 \qquad (8.6.12)$$

and after some algebra, we can rewrite Eq. (8.6.12) as

$$s = v_0 \left(\frac{v_0 - v_{x,0}}{\mu_k g} \right) - \frac{1}{2} \mu_k g \left(\frac{v_0 - v_{x,0}}{\mu_k g} \right)^2 = \frac{v_0^2}{2\mu_k g} - \frac{v_{x,0}^2}{2\mu_k g}. \qquad (8.6.13)$$

Substituting Eq. (8.6.6) for $v_{x,0}^2$ into Eq. (8.6.13), yields the distance the object traveled on the landing,

$$s = \frac{v_0^2 - (3gd^2 / 2h)}{2\mu_k g}. \qquad (8.6.14)$$

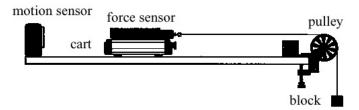**Example 8.7 Cart Moving on a Track**

Figure 8.33 A falling block will accelerate a cart on a track via the pulling force of the string. The force sensor measures the tension in the string.

Consider a cart that is free to move along a horizontal track (Figure 8.33). A force is applied to the cart via a string that is attached to a force sensor mounted on the cart, wrapped around a pulley and attached to a block on the other end. When the block is released the cart will begin to accelerate. The force sensor and cart together have a mass m_C, and the suspended block has mass m_B. You may neglect the small mass of the string and pulley, and assume the string is inextensible. The frictional force is modeled as a coefficient of kinetic friction μ_k between the cart and the track (almost all of the friction is in the wheel bearings, and the model is quite good). (a) What is the acceleration of the cart? (b) What is the tension in the string?

Solution: In general, we would like to draw free-body diagrams on all the individual objects (cart, sensor, pulley, rope, and block) but we can also choose a system consisting of two (or more) objects knowing that the forces of interaction between any two objects will cancel in pairs by Newton's Third Law. In this example, we shall choose the sensor/cart as one free-body, and the block as the other free-body. The free-body force diagram for the sensor/cart is shown in Figure 8.34.
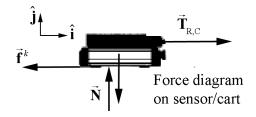
Figure 8.34 Force diagram on sensor/cart with a vector decomposition of the contact force into horizontal and vertical components

There are three forces acting on the sensor/cart: the gravitational force $m_C\vec{\mathbf{g}}$, the pulling force $\vec{\mathbf{T}}_{R,C}$ of the rope on the force sensor, and the contact force between the track and the cart. In Figure 8.34, we decompose the contact force into its two components, the kinetic frictional force $\vec{\mathbf{f}}_k = -f_k\hat{\mathbf{i}}$ and the normal force, $\vec{\mathbf{N}} = N\,\hat{\mathbf{j}}$.

The cart is only accelerating in the horizontal direction with $\vec{\mathbf{a}}_C = a_{C,x}\,\hat{\mathbf{i}}$, so the component of the force in the vertical direction must be zero, $a_{C,y} = 0$. We can now apply Newton's Second Law in the horizontal and vertical directions and find that

$$\hat{\mathbf{i}}: \ T_{R,C} - f_k = m_C a_{C,x} \tag{8.6.15}$$

$$\hat{\mathbf{j}}: \ N - m_C g = 0. \tag{8.6.16}$$

From Eq. (8.6.16), we conclude that the normal component is

$$N = m_C g. \tag{8.6.17}$$

We use Equation (8.6.17) for the normal force to find that the magnitude of the kinetic frictional force is

$$f_k = \mu_k N = \mu_k m_C g. \tag{8.6.18}$$

Then Equation (8.6.15) becomes

$$T_{R,C} - \mu_k m_C g = m_C a_{C,x}. \tag{8.6.19}$$

The force diagram for the block is shown in Figure 8.35. The two forces acting on the block are the pulling force $\vec{\mathbf{T}}_{R,B}$ of the string and the gravitational force $m_B\vec{\mathbf{g}}$. We now apply Newton's Second Law to the block and find that

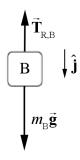$$\hat{\mathbf{j}}: m_B g - T_{R,B} = m_B a_{B,y}. \tag{8.6.20}$$

Figure 8.35 Forces acting on the block

In Equation (8.6.20), the symbol $a_{B,y}$ represents the component of the acceleration with sign determined by our choice of direction for the unit vector $\hat{\mathbf{j}}$. Note that we made a different choice of direction for the unit vector in the vertical direction in the free-body diagram for the block shown in Figure 8.34. Each free-body diagram has an independent set of unit vectors that define a sign convention for vector decomposition of the forces acting on the free-body and the acceleration of the free-body. In our example, with the unit vector pointing downwards in Figure 8.35, if we solve for the component of the acceleration and it is positive, then we know that the direction of the acceleration is downwards.

There is a second subtle way that signs are introduced with respect to the forces acting on a free-body. In our example, the force between the string and the block acting on the block points upwards, so in the vector decomposition of the forces acting on the block that appears on the left-hand side of Equation (8.6.20), this force has a minus sign and the quantity $\vec{\mathbf{T}}_{R,B} = -T_{R,B}\hat{\mathbf{j}}$ where $T_{R,B}$ is assumed positive. However, if for some reason we were uncertain about the direction of the force between the string and the block acting on the block, and drew the arrow downwards, then when we solved the problem we would discover that $T_{R,B}$ is negative, indicating that the force points in a direction opposite the direction of the arrow on the free-body diagram.

Our assumption that the mass of the rope and the mass of the pulley are negligible enables us to assert that the tension in the rope is uniform and equal in magnitude to the forces at each end of the rope,

$$T_{R,B} = T_{R,C} \equiv T . \tag{8.6.21}$$

We also assumed that the string is inextensible (does not stretch). This implies that the rope, block, and sensor/cart all have the same magnitude of acceleration,

$$a_{C,x} = a_{B,y} \equiv a . \tag{8.6.22}$$

Using Equations (8.6.21) and (8.6.22), we can now rewrite the equation of motion for the sensor/cart, Equation (8.6.19), as

$$T - \mu_k m_C g = m_C a,\tag{8.6.23}$$

and the equation of motion (8.6.20) for the block as

$$m_B g - T = m_B a.\tag{8.6.24}$$

We have only two unknowns T and a, so we can now solve the two equations (8.6.23) and (8.6.24) simultaneously for the acceleration of the sensor/cart and the tension in the rope. We first solve Equation (8.6.23) for the tension

$$T = \mu_k m_C g + m_C a\tag{8.6.25}$$

and then substitute Equation (8.6.25) into Equation (8.6.24) and find that

$$m_B g - (\mu_k m_C g + m_C a) = m_B a.\tag{8.6.26}$$

We can now solve Equation (8.6.26) for the acceleration,

$$a = \frac{m_B g - \mu_k m_C g}{m_C + m_B}.\tag{8.6.27}$$

Substitution of Equation (8.6.27) into Equation (8.6.25) gives the tension in the string,

$$
\begin{aligned}
T &= \mu_k m_C g + m_C a \\
&= \mu_k m_C g + m_C \frac{m_B g - \mu_k m_C g}{m_C + m_B} \\
&= (\mu_k + 1) \frac{m_C m_B}{m_C + m_B} g.
\end{aligned}\tag{8.6.28}
$$

In this example, we applied Newton's Second Law to two objects, one a composite object consisting of the sensor and the cart, and the other the block. We analyzed the forces acting on each object and also any constraints imposed on the acceleration of each object. We used the force laws for kinetic friction and gravitation on each free-body system. The three equations of motion enable us to determine the forces that depend on the parameters in the example: the tension in the rope, the acceleration of the objects, and normal force between the cart and the table.

Example 8.8 Pulleys and Ropes Constraint Conditions

Consider the arrangement of pulleys and blocks shown in Figure 8.36. The pulleys are assumed massless and frictionless and the connecting strings are massless and unstretchable. Denote the respective masses of the blocks as m_1, m_2 and m_3. The upper

pulley in the figure is free to rotate but its center of mass does not move. Both pulleys have the same radius R. (a) How are the accelerations of the objects related? (b) Draw force diagrams on each moving object. (c) Solve for the accelerations of the objects and the tensions in the ropes.
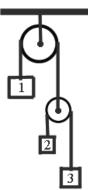

Figure 8.36 Constrained pulley system

Solution: (a) Choose an origin at the center of the upper pulley. Introduce coordinate functions for the three moving blocks, y_1, y_2 and y_3. Introduce a coordinate function y_P for the moving pulley (the pulley on the lower right in Figure 8.37). Choose downward for positive direction; the coordinate system is shown in the figure below then.
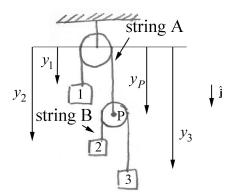

Figure 8.37 Coordinated system for pulley system

The length of string A is given by

$$l_A = y_1 + y_P + \pi R \tag{8.6.29}$$

where πR is the arc length of the rope that is in contact with the pulley. This length is constant, and so the second derivative with respect to time is zero,

$$0 = \frac{d^2 l_A}{dt^2} = \frac{d^2 y_1}{dt^2} + \frac{d^2 y_P}{dt^2} = a_{y,1} + a_{y,P} . \tag{8.6.30}$$

Thus block 1 and the moving pulley's components of acceleration are equal in magnitude but opposite in sign,

$$a_{y,P} = -a_{y,1}. \tag{8.6.31}$$

The length of string B is given by

$$l_B = (y_3 - y_P) + (y_2 - y_P) + \pi R = y_3 + y_2 - 2y_P + \pi R \tag{8.6.32}$$

where πR is the arc length of the rope that is in contact with the pulley. This length is also constant so the second derivative with respect to time is zero,

$$0 = \frac{d^2 l_B}{dt^2} = \frac{d^2 y_2}{dt^2} + \frac{d^2 y_3}{dt^2} - 2\frac{d^2 y_P}{dt^2} = a_{y,2} + a_{y,3} - 2a_{y,P}. \tag{8.6.33}$$

We can substitute Equation (8.6.31) for the pulley acceleration into Equation (8.6.33) yielding the *constraint relation* between the components of the acceleration of the three blocks,

$$0 = a_{y,2} + a_{y,3} + 2a_{y,1}. \tag{8.6.34}$$

b) Free-body Force diagrams: the forces acting on block 1 are: the gravitational force $m_1\vec{g}$ and the pulling force $\vec{T}_{A,1}$ of string A acting on the block 1. Denote the magnitude of this force by T_A. Because the string is assumed to be massless and the pulley is assumed to be massless and frictionless, the tension T_A in the string is uniform and equal in magnitude to the pulling force of the string on the block. The free-body diagram on block 1 is shown in Figure 8.38.
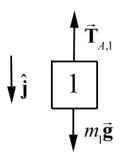

Figure 8.38 Free-body force diagram on block 1

Newton's Second Law applied to block 1 is then

$$\hat{\mathbf{j}}: \ m_1 g - T_A = m_1 a_{y,1}. \tag{8.6.35}$$

The forces on the block 2 are the gravitational force $m_2 \vec{\mathbf{g}}$ and string B holding the block, $\vec{\mathbf{T}}_{B,2}$, with magnitude T_B. The free-body diagram for the forces acting on block 2 is shown in Figure 8.39.
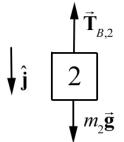

Figure 8.39 Free-body force diagram on block 2

Newton's second Law applied to block 2 is

$$\hat{\mathbf{j}}:\ m_2 g - T_B = m_2\, a_{y,2}. \tag{8.6.36}$$

The forces on the block 3 are the gravitational force $m_3 \vec{\mathbf{g}}$ and string holding the block, $\vec{\mathbf{T}}_{B,3}$, with magnitude equal to T_B because pulley P has been assumed to be both frictionless and massless. The free-body diagram for the forces acting on block 3 is shown in Figure 8.40.
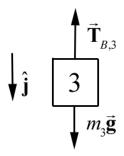

Figure 8.40 Free-body force diagram on block 3

Newton's second Law applied to block 3 is

$$\hat{\mathbf{j}}:\ m_3 g - T_B = m_3\, a_{y,3}. \tag{8.6.37}$$

The forces on the moving pulley P are the gravitational force $m_P \vec{\mathbf{g}} = \vec{\mathbf{0}}$ (the pulley is assumed massless); string B pulls down on the pulley on each side with a force, $\vec{\mathbf{T}}_{B,P}$, which has magnitude T_B. String A holds the pulley up with a force $\vec{\mathbf{T}}_{A,P}$ with the

magnitude T_A equal to the tension in string A. The free-body diagram for the forces acting on the moving pulley is shown in Figure 8.41.
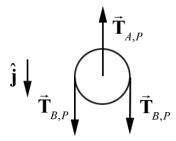

Figure 8.41 Free-body force diagram on pulley

Newton's second Law applied to the pulley is

$$\hat{\mathbf{j}}: \ 2T_B - T_A = m_P\, a_{y,P} = 0\,. \tag{8.6.38}$$

Because the pulley is assumed to be massless, we can use this last equation to determine the condition that the tension in the two strings must satisfy,

$$2T_B = T_A \tag{8.6.39}$$

We are now in position to determine the accelerations of the blocks and the tension in the two strings. We record the relevant equations as a summary.

$$0 = a_{y,2} + a_{y,3} + 2a_{y,1} \tag{8.6.40}$$
$$m_1 g - T_A = m_1\, a_{y,1} \tag{8.6.41}$$
$$m_2 g - T_B = m_2\, a_{y,2} \tag{8.6.42}$$
$$m_3 g - T_B = m_3\, a_{y,3} \tag{8.6.43}$$
$$2T_B = T_A\,. \tag{8.6.44}$$

There are five equations with five unknowns, so we can solve this system. We shall first use Equation (8.6.44) to eliminate the tension T_A in Equation (8.6.41), yielding

$$m_1 g - 2T_B = m_1\, a_{y,1}\,. \tag{8.6.45}$$

We now solve Equations (8.6.42), (8.6.43) and (8.6.45) for the accelerations,

$$a_{y,2} = g - \frac{T_B}{m_2} \tag{8.6.46}$$

$$a_{y,3} = g - \frac{T_B}{m_3} \tag{8.6.47}$$

$$a_{y,1} = g - \frac{2T_B}{m_1}. \tag{8.6.48}$$

We now substitute these results for the accelerations into the constraint equation, Equation (8.6.40),

$$0 = g - \frac{T_B}{m_2} + g - \frac{T_B}{m_3} + 2g - \frac{4T_B}{m_1} = 4g - T_B\left(\frac{1}{m_2} + \frac{1}{m_3} + \frac{4}{m_1}\right). \tag{8.6.49}$$

We can now solve this last equation for the tension in string B,

$$T_B = \frac{4g}{\left(\dfrac{1}{m_2} + \dfrac{1}{m_3} + \dfrac{4}{m_1}\right)} = \frac{4g\, m_1\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}. \tag{8.6.50}$$

From Equation (8.6.44), the tension in string A is

$$T_A = 2T_B = \frac{8g\, m_1\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}. \tag{8.6.51}$$

We find the acceleration of block 1 from Equation (8.6.48), using Equation (8.6.50) for the tension in string B,

$$a_{y,1} = g - \frac{2T_B}{m_1} = g - \frac{8g\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3} = g\, \frac{m_1\, m_3 + m_1\, m_2 - 4\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}. \tag{8.6.52}$$

We find the acceleration of block 2 from Equation (8.6.46), using Equation (8.6.50) for the tension in string B,

$$a_{y,2} = g - \frac{T_B}{m_2} = g - \frac{4g\, m_1\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3} = g\, \frac{-3\, m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}. \tag{8.6.53}$$

Similarly, we find the acceleration of block 3 from Equation (8.6.47), using Equation (8.6.50) for the tension in string B,

$$a_{y,3} = g - \frac{T_B}{m_3} = g - \frac{4g\, m_1\, m_2}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3} = g\, \frac{m_1\, m_3 - 3\, m_1\, m_2 + 4\, m_2\, m_3}{m_1\, m_3 + m_1\, m_2 + 4\, m_2\, m_3}. \tag{8.6.54}$$

As a check on our algebra we note that

$$2a_{1,y} + a_{2,y} + a_{3,y} =$$

$$2g\frac{m_1 m_3 + m_1 m_2 - 4m_2 m_3}{m_1 m_3 + m_1 m_2 + 4m_2 m_3} + g\frac{-3m_1 m_3 + m_1 m_2 + 4m_2 m_3}{m_1 m_3 + m_1 m_2 + 4m_2 m_3} + g\frac{m_1 m_3 - 3m_1 m_2 + 4m_2 m_3}{m_1 m_3 + m_1 m_2 + 4m_2 m_3}$$

$$= 0.$$

Example 8.9 Accelerating Wedge

A 45° wedge is pushed along a table with constant acceleration $\vec{\mathbf{A}}$ according to an observer at rest with respect to the table. A block of mass m slides without friction down the wedge (Figure 8.42). Find its acceleration with respect to an observer at rest with respect to the table. Write down a plan for finding the magnitude of the acceleration of the block. Make sure you clearly state which concepts you plan to use to calculate any relevant physical quantities. Also clearly state any assumptions you make. Be sure you include any free-body force diagrams or sketches that you plan to use.
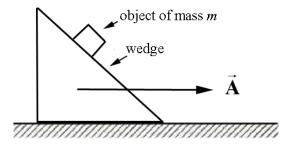

Figure 8.42 Block on accelerating wedge

Solution: Choose a coordinate system for the block and wedge as shown in Figure 8.43. Then $\vec{\mathbf{A}} = A_{x,w}\,\hat{\mathbf{i}}$ where $A_{x,w}$ is the x-component of the acceleration of the wedge.
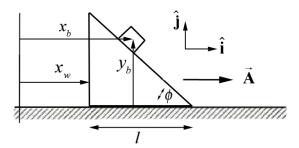

Figure 8.43 Coordinate system for block on accelerating wedge

We shall apply Newton's Second Law to the block sliding down the wedge. Because the wedge is accelerating, there is a constraint relation between the x- and y- components of the acceleration of the block. In order to find that constraint we choose a coordinate system for the wedge and block sliding down the wedge shown in the figure below. We shall find the constraint relationship between the components of the accelerations of the block and wedge by a geometric argument. From the figure above, we see that

$$\tan\phi = \frac{y_b}{l-(x_b-x_w)}. \tag{8.6.55}$$

Therefore

$$y_b = (l-(x_b-x_w))\tan\phi. \tag{8.6.56}$$

If we differentiate Eq. (8.6.56) twice with respect to time noting that

$$\frac{d^2l}{dt^2} = 0 \tag{8.6.57}$$

we have that

$$\frac{d^2y_b}{dt^2} = -\left(\frac{d^2x_b}{dt^2} - \frac{d^2x_w}{dt^2}\right)\tan\phi. \tag{8.6.58}$$

Therefore

$$a_{b,y} = -(a_{b,x} - A_{x,w})\tan\phi \tag{8.6.59}$$

recalling that

$$A_{x,w} = \frac{d^2x_w}{dt^2}. \tag{8.6.60}$$

We now draw a free-body force diagram for the block (Figure 8.44). Newton's Second Law in the $\hat{\mathbf{i}}$ - direction becomes

$$N\sin\phi = ma_{b,x}. \tag{8.6.61}$$

and the $\hat{\mathbf{j}}$-direction becomes

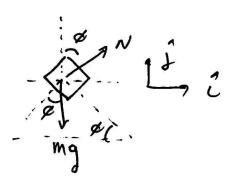$$N\cos\phi - mg = ma_{b,y} \tag{8.6.62}$$

Figure 8.44 Free-body force diagram on block

We can solve for the normal force from Eq. (8.6.61):

$$N = \frac{ma_{b,x}}{\sin\phi} \tag{8.6.63}$$

We now substitute Eq. (8.6.59) and Eq. (8.6.63) into Eq. (8.6.62) yielding

$$\frac{ma_{b,x}}{\sin\phi}\cos\phi - mg = m(-(a_{b,x} - A_{w,x})\tan\phi). \qquad (8.6.64)$$

We now clean this up yielding

$$ma_{b,x}(\cotan\phi + \tan\phi) = m(g + A_{w,x}\tan\phi) \qquad (8.6.65)$$

Thus the x-component of the acceleration is then

$$a_{b,x} = \frac{g + A_{w,x}\tan\phi}{\cotan\phi + \tan\phi} \qquad (8.6.66)$$

From Eq. (8.6.59), the y-component of the acceleration is then

$$a_{b,y} = -(a_{b,x} - A_{w,x})\tan\phi = -\left(\frac{g + A_{w,x}\tan\phi}{\cotan\phi + \tan\phi} - A_{w,x}\right)\tan\phi. \qquad (8.6.67)$$

This simplifies to

$$a_{b,y} = \frac{A_{w,x} - g\tan\phi}{\cotan\phi + \tan\phi} \qquad (8.6.68)$$

When $\phi = 45°$, $\cotan 45° = \tan 45° = 1$, and so Eq. (8.6.66) becomes

$$a_{b,x} = \frac{g + A_{w,x}}{2} \qquad (8.6.69)$$

and Eq. (8.6.68) becomes

$$a_{b,y} = \frac{A - g}{2}. \qquad (8.6.70)$$

The magnitude of the acceleration is then

$$a = \sqrt{a_{b,x}^2 + a_{b,y}^2} = \sqrt{\left(\frac{g + A_{w,x}}{2}\right)^2 + \left(\frac{A_{w,x} - g}{2}\right)^2} \qquad (8.6.71)$$

$$a = \sqrt{\left(\frac{g^2 + A_{w,x}^2}{2}\right)}.$$

Example 8.10: Capstan

A device called a capstan is used aboard ships in order to control a rope that is under great tension. The rope is wrapped around a fixed drum of radius R, usually for several turns (Figure 8.45 shows about three fourths turn as seen from overhead). The load on the rope pulls it with a force T_A, and the sailor holds the other end of the rope with a much smaller force T_B. The coefficient of static friction between the rope and the drum is μ_s. The sailor is holding the rope so that it is just about to slip. Show that $T_B = T_A e^{-\mu_s \theta_{BA}}$, where θ_{BA} is the angle subtended by the rope on the drum.
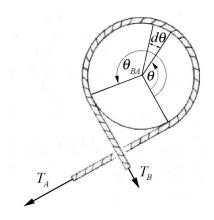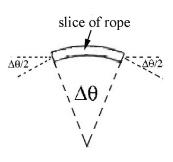

Figure 8.45 Capstan

Figure 8.46 Small slice of rope

Solution: We begin by considering a small slice of rope of arc length $R\Delta\theta$, shown in the Figure 8.46. We choose unit vectors for the force diagram on this section of the rope and indicate them on Figure 8.47. The right edge of the slice is at angle θ and the left edge of the slice is at $\theta + \Delta\theta$. The angle edge end of the slice makes with the horizontal is $\Delta\theta / 2$. There are four forces acting on this section of the rope. The forces are the normal force between the capstan and the rope pointing outward, a static frictional force and the tensions at either end of the slice. The rope is held at the just slipping point, so if the load exerts a greater force the rope will slip to the right. Therefore the direction of the static frictional force between the capstan and the rope, acting on the rope, points to the left. The tension on the right side of the slice is denoted by $T(\theta) \equiv T$, while the tension on the left side of the slice is denoted by $T(\theta + \Delta\theta) \equiv T + \Delta T$. Does the tension in this slice from the right side to the left, increase, remain the same, or decrease? The tension decreases because the load on the left side is less than the load on the right side. Note that $\Delta T < 0$.
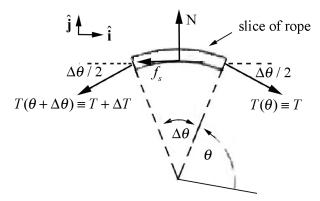
Figure 8.47 Free-body force diagram on small slice of rope

The vector decomposition of the forces is given by

$$\hat{\mathbf{i}}: T\cos(\Delta\theta/2) - f_s - (T + \Delta T)\cos(\Delta\theta/2) \qquad (8.6.72)$$

$$\hat{\mathbf{j}}: -T\sin(\Delta\theta/2) + N - (T + \Delta T)\sin(\Delta\theta/2). \qquad (8.6.73)$$

For small angles $\Delta\theta$, $\cos(\Delta\theta/2) \cong 1$ and $\sin(\Delta\theta/2) \cong \Delta\theta/2$. Using the small angle approximations, the vector decomposition of the forces in the x-direction (the $+\hat{\mathbf{i}}$-direction) becomes

$$T\cos(\Delta\theta/2) - f_s - (T + \Delta T)\cos(\Delta\theta/2) \simeq T - f_s - (T + \Delta T)$$
$$= -f_s - \Delta T \qquad . \qquad (8.6.74)$$

By the static equilibrium condition the sum of the x-components of the forces is zero,

$$-f_s - \Delta T = 0. \qquad (8.6.75)$$

The vector decomposition of the forces in the y-direction (the $+\hat{\mathbf{j}}$-direction) is approximately

$$-T\sin(\Delta\theta/2) + N - (T + \Delta T)\sin(\Delta\theta/2) \simeq -T\Delta\theta/2 + N - (T + \Delta T)\Delta\theta/2$$
$$= -T\Delta\theta + N - \Delta T\Delta\theta/2 \qquad . \qquad (8.6.76)$$

In the last equation above we can ignore the terms proportional to $\Delta T \Delta\theta$ because these are the product of two small quantities and hence are much smaller than the terms proportional to either ΔT or $\Delta\theta$. The vector decomposition in the y-direction becomes

$$-T\Delta\theta + N. \qquad (8.6.77)$$

Static equilibrium implies that this sum of the y-components of the forces is zero,

$$-T\Delta\theta + N = 0. \tag{8.6.78}$$

We can solve this equation for the magnitude of the normal force

$$N = T\Delta\theta. \tag{8.6.79}$$

The just slipping condition is that the magnitude of the static friction attains its maximum value

$$f_s = (f_s)_{\text{max}} = \mu_s N. \tag{8.6.80}$$

We can now combine the Equations (8.6.75) and (8.6.80) to yield

$$\Delta T = -\mu_s N. \tag{8.6.81}$$

Now substitute the magnitude of the normal force, Equation (8.6.79), into Equation (8.6.81), yielding

$$-\mu_s T\Delta\theta - \Delta T = 0. \tag{8.6.82}$$

Finally, solve this equation for the ratio of the change in tension to the change in angle,

$$\frac{\Delta T}{\Delta\theta} = -\mu_s T. \tag{8.6.83}$$

The derivative of tension with respect to the angle θ is defined to be the limit

$$\frac{dT}{d\theta} \equiv \lim_{\Delta\theta \to 0} \frac{\Delta T}{\Delta\theta}, \tag{8.6.84}$$

and Equation (8.6.83) becomes

$$\frac{dT}{d\theta} = -\mu_s T. \tag{8.6.85}$$

This is an example of a first order linear differential equation that shows that the rate of change of tension with respect to the angle θ is proportional to the negative of the tension at that angle θ. This equation can be solved by integration using the technique of separation of variables. We first rewrite Equation (8.6.85) as

$$\frac{dT}{T} = -\mu_s d\theta. \tag{8.6.86}$$

Integrate both sides, noting that when $\theta = 0$, the tension is equal to force of the load T_A, and when angle $\theta = \theta_{A,B}$ the tension is equal to the force T_B the sailor applies to the rope,

$$\int_{T=T_A}^{T=T_B} \frac{dT}{T} = -\int_{\theta=0}^{\theta=\theta_{BA}} \mu_s\, d\theta . \tag{8.6.87}$$

The result of the integration is

$$\ln\left(\frac{T_B}{T_A}\right) = -\mu_s \theta_{BA} . \tag{8.6.88}$$

Note that the exponential of the natural logarithm

$$\exp\left(\ln\left(\frac{T_B}{T_A}\right)\right) = \frac{T_B}{T_A}, \tag{8.6.89}$$

so exponentiating both sides of Equation (8.6.88) yields

$$\frac{T_B}{T_A} = e^{-\mu_s \theta_{BA}} ; \tag{8.6.90}$$

the tension decreases exponentially,

$$T_B = T_A e^{-\mu_s \theta_{BA}} , \tag{8.6.91}$$

Because the tension decreases exponentially, the sailor need only apply a small force to prevent the rope from slipping.