Chapter 12 Momentum and the Flow of Mass

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Even though the release was pulled, the rocket did not rise at first, but the flame came out, and there was a steady roar. After a number of seconds it rose, slowly until it cleared the flame, and then at express-train speed, curving over to the left, and striking the ice and snow, still going at a rapid rate. It looked almost magical as it rose, without any appreciably greater noise or flame, as if it said, "I've been here long enough; I think I'll be going somewhere else, if you don't mind." [1]

Robert Goddard

Preface The Challenger Flight

When the Rogers Commission in 1986 investigated the Challenger Flight disaster, a commission member, physicist Richard Feynman, made an extraordinary demonstration during the hearings.

"He (Feynman) also learned that rubber used to seal the solid rocket booster joints using O-rings, failed to expand when the temperature was at or below 32 degrees F (0 degrees C). The temperature at the time of the Challenger liftoff was 32 degrees F. Feynman now believed that he had the solution, but to test it, he dropped a piece of the O-ring material, squeezed with a C-clamp to simulate the actual conditions of the shuttle, into a glass of ice water. Ice, of course, is 32 degrees F. At this point one needs to understand exactly what role the O-rings play in the solid rocket booster (SRB) joints. When the material in the SRB start to heat up, it expands and pushes against the sides of the SRB. If there is an opening in a joint in the SRB, the gas tries to escape through that opening (think of it like water in a tea kettle escaping through the spout.) This leak in the Challenger's SRB was easily visible as a small flicker in a launch photo. This flicker turned into a flame and began heating the fuel tank, which then ruptured. When this happened, the fuel tank released liquid hydrogen into the atmosphere where it exploded. As Feynman explained, because the O-rings cannot expand in 32 degree weather, the gas finds gaps in the joints, which led to the explosion of the booster and then the shuttle itself."[2]

In the Report of the Presidential Commission on the Space Shuttle Challenger Accident (1986), *Appendix F - Personal observations on the reliability of the Shuttle*, Feynman wrote

The Challenger flight is an excellent example. … The O-rings of the Solid Rocket Boosters were not designed to erode. Erosion was a clue that something was wrong. Erosion was not something from which safety can be inferred. There was no way, without full understanding, that one could have confidence that conditions the next time might not

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[1] describing the first rocket flight using liquid propellants at Aunt Effie's farm, 17 March 1926.

[2] http://www.fotuva.org/online/frameload.htm?/online/challenger.htm.

produce erosion three times more severe than the time before. Nevertheless, officials fooled themselves into thinking they had such understanding and confidence, in spite of the peculiar variations from case to case. A mathematical model was made to calculate erosion. This was a model based not on physical understanding but on empirical curve fitting. To be more detailed, it was supposed a stream of hot gas impinged on the O-ring material, and the heat was determined at the point of stagnation (so far, with reasonable physical, thermodynamic laws). But to determine how much rubber eroded it was assumed this depended only on this heat by a formula suggested by data on a similar material. A logarithmic plot suggested a straight line, so it was supposed that the erosion varied as the .58 power of the heat, the .58 being determined by a nearest fit. At any rate, adjusting some other numbers, it was determined that the model agreed with the erosion (to depth of one-third the radius of the ring). There is nothing much so wrong with this as believing the answer! Uncertainties appear everywhere. How strong the gas stream might be was unpredictable, it depended on holes formed in the putty. Blow-by showed that the ring might fail even though not, or only partially eroded through. The empirical formula was known to be uncertain, for it did not go directly through the very data points by which it was determined. There were a cloud of points some twice above, and some twice below the fitted curve, so erosions twice predicted were reasonable from that cause alone. Similar uncertainties surrounded the other constants in the formula, etc., etc. When using a mathematical model careful attention must be given to uncertainties in the model. …

In any event this has had very unfortunate consequences, the most serious of which is to encourage ordinary citizens to fly in such a dangerous machine, as if it had attained the safety of an ordinary airliner. The astronauts, like test pilots, should know their risks, and we honor them for their courage. Who can doubt that McAuliffe was equally a person of great courage, who was closer to an awareness of the true risk than NASA management would have us believe? Let us make recommendations to ensure that NASA officials deal in a world of reality in understanding technological weaknesses and imperfections well enough to be actively trying to eliminate them. …. For a successful technology, reality must take precedence over public relations, for nature cannot be fooled.[3]

12.1 Introduction

So far we have restricted ourselves to considering systems consisting of discrete objects or point-like objects that have fixed amounts of mass. We shall now consider systems in which material flows between the objects in the system, for example we shall consider coal falling from a hopper into a moving railroad car, sand leaking from railroad car fuel, grain moving forward into a railroad car, and fuel ejected from the back of a rocket, In each of these examples material is continuously flows into or out of an object. We have already shown that the total external force causes the momentum of a system to change,

[3] R. P. Feynman, *Appendix F - Personal observations on the reliability of the Shuttle*, Report of the PRESIDENTIAL COMMISSION on the Space Shuttle Challenger Accident (1986), http://history.nasa.gov/rogersrep/genindex.htm.

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\vec{\mathbf{p}}_{\text{system}}}{dt}. \tag{12.1.1}$$

We shall analyze how the momentum of the constituent elements our system change over a time interval $[t, t+\Delta t]$, and then consider the limit as $\Delta t \to 0$. We can then explicit calculate the derivative on the right hand side of Eq. (12.1.1) and Eq. (12.1.1) becomes

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\vec{\mathbf{p}}_{\text{system}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{p}}_{\text{system}}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{p}}_{\text{system}}(t+\Delta t) - \vec{\mathbf{p}}_{\text{system}}(t)}{\Delta t}. \tag{12.1.2}$$

We need to be very careful how we apply this generalized version of Newton's Second Law to systems in which mass flows between constituent objects. In particular, when we isolate elements as part of our system we must be careful to identify the mass Δm of the material that continuous flows in or out of an object that is part of our system during the time interval Δt under consideration.

We shall consider four categories of mass flow problems that are characterized by the momentum transfer of the material of mass Δm.

12.1.1 Transfer of Material into an Object, but no Transfer of Momentum

Consider for example rain falling vertically downward with speed u into cart of mass m moving forward with speed v. A small amount of falling rain Δm_r has no component of momentum in the direction of motion of the cart. There is a transfer of rain into the cart but no transfer of momentum in the direction of motion of the cart (Figure 12.1).

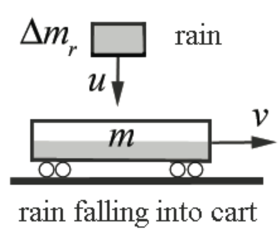


Figure 12.1 Transfer of rain mass into the cart but no transfer of momentum in direction of motion

12.1.2 Transfer of Material Out of an Object, but no Transfer of Momentum

The material continually leaves the object but it does not transport any momentum away from the object in the direction of motion of the object (Figure 12.2). Consider an ice skater gliding on ice at speed v holding a bag of sand that is leaking straight down with respect to the moving skater. The sand continually leaves the bag but it does not transport any momentum away from the bag in the direction of motion of the object. In Figure 12.2, sand of mass Δm_s leaves the bag.

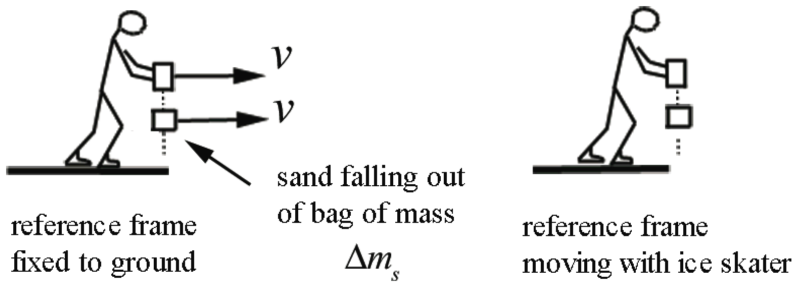


Figure 12.2 Transfer of mass out of object but no transfer of momentum in direction of motion

12.1.3 Transfer of Material Impulses Object Via Transfer of Momentum

Suppose a fire hose is used to put out a fire on a boat. The incoming water with speed u continually hits the boat impulsing it forward. Figure 12.3 shows a column of water of mass Δm_s approaching a boat of mass m_b that is moving forward with speed v.

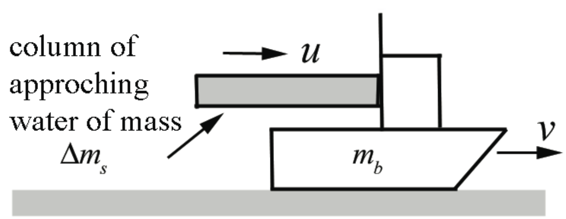


Figure 12.3 Transfer of mass provides impulse on object

12.1.4 Material Continually Ejected From Object results in Recoil of Object

When fuel of mass Δm_f is ejected from the back of a rocket with speed u relative to the rocket, the rocket of mass m_r recoils forward. Figure 12.4a shows the recoil of the rocket in the reference frame of the rocket. The rocket recoils forward with speed Δv_r. In a reference frame in which the rocket is moving forward with speed v_r, then the speed after recoil is $v_r + \Delta v_r$. The speed of the backwardly ejected fuel is $u - v_r$ (Figure 12.4b).

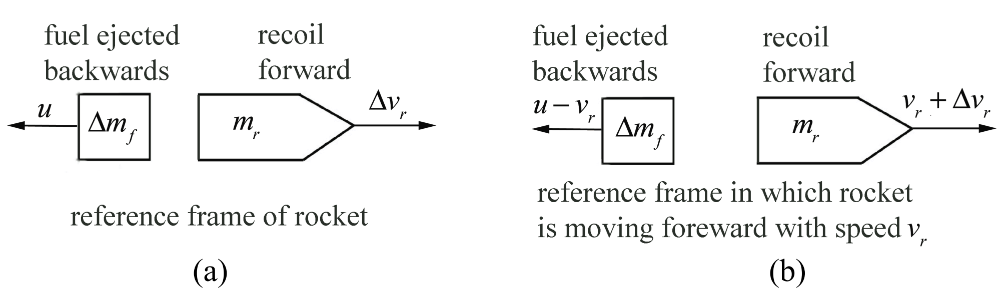


Figure 12.4 Transfer of mass out of rocket provides impulse on rocket in (a) reference frame of rocket, (b) reference frame in which rocket moves with speed v_r

We must carefully identify the momentum of the object and the material transferred at time t in order to determine $\vec{\mathbf{p}}_{\text{system}}(t)$. We must also identify the momentum of the object and the material transferred at time $t+\Delta t$ in order to determine $\vec{\mathbf{p}}_{\text{system}}(t+\Delta t)$ as well. Recall that when we defined the momentum of a system, we assumed that the mass of the system remain constant. Therefore we cannot ignore the momentum of the transferred material at time $t+\Delta t$ even though it may have left the object; it is still part of our system (or at time t even though it has not flowed into the object yet).

12.2 Worked Examples

Example 12.1 Filling a Coal Car

An empty coal car of mass m_0 starts from rest under an applied force of magnitude F. At the same time coal begins to run into the car at a steady rate b from a coal hopper at rest along the track (Figure 12.5). Find the speed when a mass m_c of coal has been transferred.

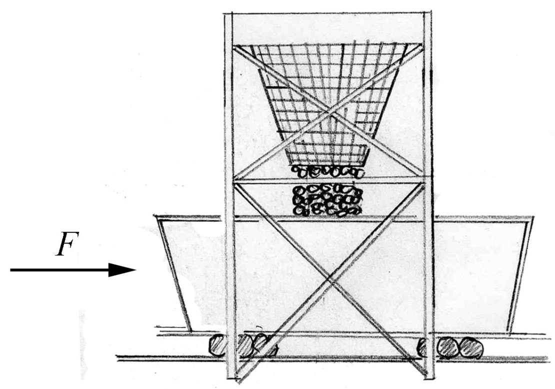


Figure 12.5 Filling a coal car

Solution: We shall analyze the momentum changes in the horizontal direction, which we call the x-direction. Because the falling coal does not have any horizontal velocity, the falling coal is not transferring any momentum in the x-direction to the coal car. So we shall take as our system the empty coal car and a mass m_c of coal that has been transferred. Our initial state at $t=0$ is when the coal car is empty and at rest before any coal has been transferred. The x-component of the momentum of this initial state is zero,

$$p_x(0)=0. \tag{12.1.3}$$

Our final state at $t=t_f$ is when all the coal of mass $m_c = bt_f$ has been transferred into the car that is now moving at speed v_f. The x-component of the momentum of this final state is

$$p_x(t_f) = (m_0 + m_c)v_f = (m_0 + bt_f)v_f \,. \quad (12.1.4)$$

There is an external constant force $F_x = F$ applied through the transfer. The momentum principle applied to the x-direction is

$$\int_0^{t_f} F_x dt = \Delta p_x = p_x(t_f) - p_x(0) \,. \quad (12.1.5)$$

Because the force is constant, the integral is simple and the momentum principle becomes

$$Ft_f = (m_0 + bt_f)v_f \,. \quad (12.1.6)$$

So the final speed is

$$v_f = \frac{Ft_f}{(m_0 + bt_f)} \,. \quad (12.1.7)$$

Example 12.2 Emptying a Freight Car

A freight car of mass m_c contains a mass of sand m_s. At $t = 0$ a constant horizontal force of magnitude F is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at the constant rate $b = dm_s / dt$. Find the speed of the freight car when all the sand is gone (Figure 12.6). Assume that the freight car is at rest at $t = 0$.

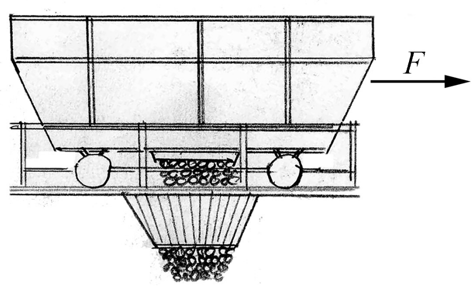
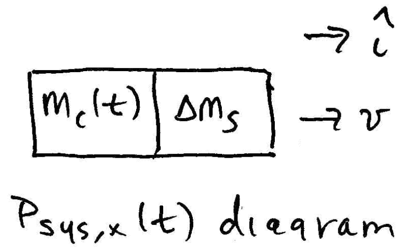


Figure 12.6 Emptying a freight car

Solution: Choose the positive x-direction to point in the direction that the car is moving. Let's take as our system the amount of sand of mass Δm_s that leaves the freight car during the time interval $[t, t + \Delta t]$, and the freight car and whatever sand is in it at time t. The momentum diagram for the system at time t is shown in Figure 12.7.



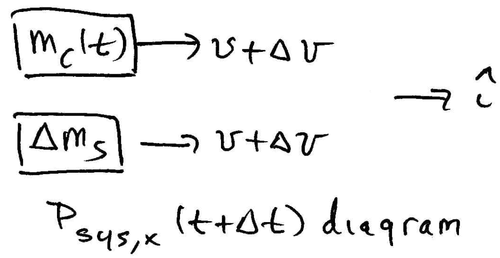


Figure 12.7 Momentum diagram at time t **Figure 12.8** Momentum diagram at time $t+\Delta t$

At the beginning of the interval the car and sand are moving with speed v so the x-component of the momentum at time t is given by

$$p_{sys,x}(t) = (\Delta m_s + m_c(t))v), \qquad (12.1.8)$$

where $m_c(t)$ is the mass of the car and sand in it at time t. The momentum diagram for the system at time $t+\Delta t$ is shown in Figure 12.8.

Note that the sand that leaves the car is shown with speed $v+\Delta v$. This implies that all the sand leaves the car with the speed of the car at the end of the interval. This is an approximation. Because the sand leaves continuous, the speed will vary from v to $v+\Delta v$. As we will shortly see, the momentum of the sand $p_{s,x}(t+\Delta t) = \Delta m_s(v+\Delta v)$ is approximately $\Delta m_s v$ because the additional term $\Delta m_s \Delta v$ is a "second-order" term, the product of two infinitesimal terms and hence will vanish when we take the limit that $\Delta t \to 0$. Therefore the x-component of the momentum at time $t+\Delta t$ is given by

$$p_{sys,x}(t+\Delta t) = (\Delta m_s + m_c(t))(v+\Delta v). \qquad (12.1.9)$$

We now construct a diagram that illustrates the changes in the x-component of momentum for each element of our system by subtracting Eq.(12.1.8) from Eq. (12.1.9), (Figure 12.9).

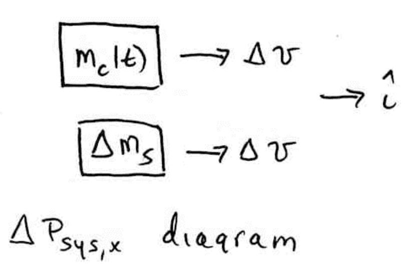


Figure 12.9 Change in momentum diagram

The change in the x-component of momentum of the system is then

$$\Delta p_{sys,x} = p_{sys,x}(t+\Delta t) - p_{sys,x}(t) = \Delta m_s \Delta v + m_c(t)\Delta v \simeq m_c(t)\Delta v \,. \quad (12.1.10)$$

In terms of the individual elements we have that $\Delta p_{sys,x} = \Delta p_{c,x} + \Delta p_{s,x}$ where

$$\Delta p_{c,x} = m_c(t)\Delta v \,, \quad (12.1.11)$$

$$\Delta p_{s,x} = \Delta m_s \Delta v \,. \quad (12.1.12)$$

Throughout the interval a constant force F is applied to the car so

$$F = \lim_{\Delta t \to 0} \frac{p_{sys,x}(t+\Delta t) - p_{sys,x}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p_{c,x}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta p_{s,x}}{\Delta t} \,. \quad (12.1.13)$$

From our analysis above, (Eqs. (12.1.11) and (12.1.12)). Eq. (12.1.13) becomes

$$F = \lim_{\Delta t \to 0} m_c(t) \frac{\Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_s \Delta v}{\Delta t} \,. \quad (12.1.14)$$

The second term vanishes when we take $\Delta t \to 0$ because it is of second order in the infinitesimal quantities (in this case $\Delta m_s \Delta v$) and so when dividing by Δt the quantity is of first order and hence vanishes since both $\Delta m_s \to 0$ and $\Delta v \to 0$. Based on this elimination of the second order term, our diagram for the changes in the x-component of momentum for each element of our system simplifies to

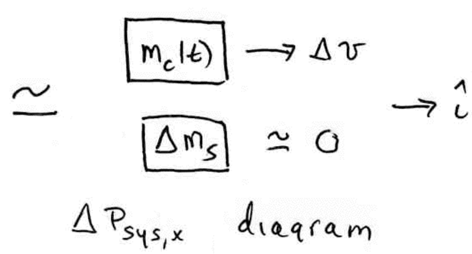


Figure 12.10 Change in momentum diagram in limit as $\Delta m_s \to 0$ and $\Delta v \to 0$

Eq. (12.1.14) becomes the differential equation

$$F = m_c(t) \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = m_c(t) \frac{dv}{dt} \,. \quad (12.1.15)$$

Denote by $m_{c,0} = m_c + m_s$ where m_c is the mass of the car and m_s is the mass of the sand in the car at $t = 0$, and $m_s(t) = bt$ is the mass of the sand that has left the car at time t,

$$m_s(t) = \int_0^t \frac{dm_s}{dt} dt = \int_0^t b dt = bt \,. \tag{12.1.16}$$

Thus

$$m_c(t) = m_{c,0} - bt = m_c + m_s - bt \,. \tag{12.1.17}$$

Using Eq. (12.1.17) we have

$$F = (m_c + m_s - bt)\frac{dv}{dt} \,. \tag{12.1.18}$$

(b) We can integrate this equation through the separation of variable technique. Rewrite Eq. (12.1.18) as

$$dv = \frac{Fdt}{(m_c + m_s - bt)} \,. \tag{12.1.19}$$

We can then integrate both sides of Eq. (12.1.19) with the limits as shown

$$\int_{v'=0}^{v'=v(t)} dv' = \int_{t'=0}^{t'=t} \frac{Fdt'}{m_c + m_s - bt'} \,. \tag{12.1.20}$$

Integration yields the x-component of the velocity of the car as a function of time

$$v(t) = -\frac{F}{b}\ln(m_c + m_s - bt')\Big|_{t'=0}^{t'=t} = -\frac{F}{b}\ln\left(\frac{m_c + m_s - bt}{m_c + m_s}\right) = \frac{F}{b}\ln\left(\frac{m_c + m_s}{m_c + m_s - bt}\right). \tag{12.1.21}$$

In writing Eq. (12.1.21), we used the property that $\ln(a) - \ln(b) = \ln(a/b)$ and consequently $\ln(a/b) = -\ln(b/a)$. Note that $m_c + m_s \geq m_c + m_s - bt$, so the term $\ln\left(\frac{m_c + m_s}{m_c + m_s - bt}\right) \geq 0$, and the x-component of the velocity of the car increases as we expect.

Example 12.3 Filling a Freight Car

Material is blown into cart A from cart B at a rate of b kilograms per second. The material leaves the chute vertically downward, so that it has the same horizontal velocity, u as cart B, (Figure 12.11). At the moment of interest, cart A has mass m_A and speed v. (a) Define the objects that will constitute your system. (b) Based on momentum flow diagrams, derive a differential equation for the velocity v.

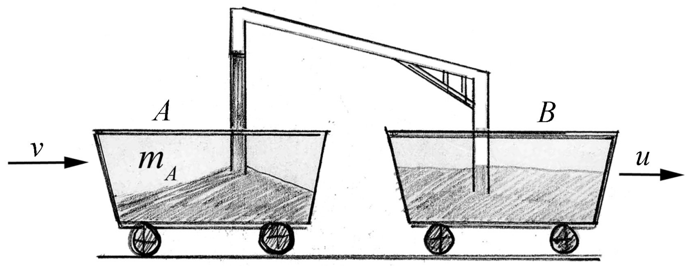
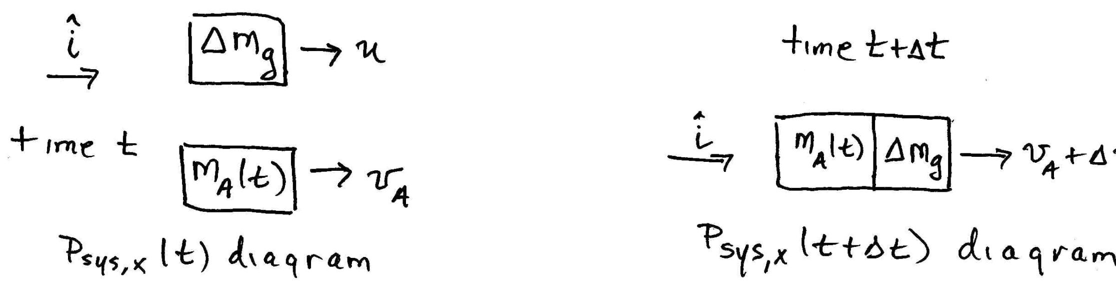


Figure 12.11 Filling a freight car

Solution: Choose positive x-direction to the right in the figure below. Define the system at time t to be the cart with whatever material is in it and the material blown into cart A during the time interval $[t, t+\Delta t]$. Denote the mass of the cart and material at time t by $m_A(t)$ and let Δm_g denote the material blown into cart A during the time interval $[t, t+\Delta t]$, with x-component of the velocity u. At time t, cart A is moving with x-component of the velocity v_A. At time $t+\Delta t$, Cart A is moving with x-component of the velocity $v_A + \Delta v_A$. The momentum diagram for time t is shown in Figure 12.12a and for $t+\Delta t$ is shown in Figure 12.12b.



(a) (b)

Figure 12.12 Momentum diagram at (a) time t and, (b) $t+\Delta t$

The diagram representing the change in the x-component of the momentum is shown in Figure 12.13.

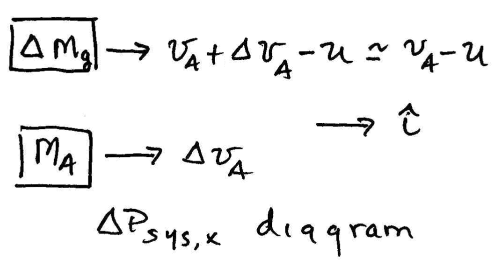


Figure 12.13 Change in momentum diagram

There are no external forces in the x-direction acting on the system, $F_{ext,x} = 0$, so the momentum principle becomes

$$0 = \Delta p_{sys,x} = \Delta p_{A,x} + \Delta p_{g,x} \,. \tag{12.1.22}$$

From our diagram showing the change in the x-component of the momentum of the elements of the system, Eq. (12.1.22) becomes

$$0 = m_A \Delta v_A + \Delta m_g (v_A - u) \,, \tag{12.1.23}$$

where we can ignore the contribution form the second order term $\Delta m \Delta v_A$. Because the cart's mass is increasing due to the material entering we have that

$$\Delta m_g = \Delta m_A \,. \tag{12.1.24}$$

and so Eq. (12.1.23) can now be written after taking limits as $\Delta t \to 0$

$$m_A dv_A = dm_A (u - v_A) \,. \tag{12.1.25}$$

We can divide both sides of Eq. (12.1.25) by dt yielding

$$m_A \frac{dv_A}{dt} = \frac{dm_A}{dt}(u - v_A) \,. \tag{12.1.26}$$

Rearranging terms and using the fact that the material is blown into the cart at a constant rate $b \equiv dm_A / dt$, we have that the rate of change of the x-component of the velocity of the cart is given by

$$\frac{dv_A}{dt} = \frac{b(u - v_A)}{m_A} \,. \tag{12.1.27}$$

We cannot directly integrate Eq. (12.1.27) with respect to dt because the mass of the cart is a function of time. In order to find the x-component of the velocity of the cart we need to know the relationship between the mass of the cart and the x-component of the velocity of the cart. There are two approaches. In the first approach we separate variables in Eq. (12.1.25)

$$\frac{dv_A}{u - v_A} = \frac{dm_A}{m_A} \,, \tag{12.1.28}$$

and then integrate

$$\int_{v'_A=0}^{v'_A=v_A(t)} \frac{dv'_A}{u - v'_A} = \int_{m'_A=m_{A,0}}^{m'_A=m_A(t)} \frac{dm'_A}{m'_A} \,, \tag{12.1.29}$$

where $m_{A,0}$ is the mass of the cart before any material has been blown in. After integration we have that

$$\ln\frac{u}{u-v_A(t)} = \ln\frac{m_A(t)}{m_{A,0}}. \tag{12.1.30}$$

Exponentiate both side gives

$$\frac{u}{u-v_A(t)} = \frac{m_A(t)}{m_{A,0}}. \tag{12.1.31}$$

We can solve this equation for the x-component of the velocity of the cart

$$v_A(t) = \frac{m_A(t) - m_{A,0}}{m_A(t)}u. \tag{12.1.32}$$

Because the material is blown into the cart at a constant rate $b \equiv dm_A / dt$, the mass of the cart as a function of time is given by

$$m_A(t) = m_{A,0} + bt. \tag{12.1.33}$$

Therefore substituting Eq. (12.1.33) into Eq. (12.1.32) yields the x-component of the velocity of the cart as a function of time

$$v_A(t) = \frac{bt}{m_{A,0} + bt}u. \tag{12.1.34}$$

In the second approach, we substitute Eq. (12.1.33) into Eq. (12.1.27) yielding

$$\frac{dv_A}{dt} = \frac{b(u - v_A)}{m_{A,0} + bt}. \tag{12.1.35}$$

We can now separate variables

$$\frac{dv_A}{u - v_A} = \frac{bdt}{m_{A,0} + bt}. \tag{12.1.36}$$

Now we can integrate

$$\int_{v_A'=0}^{v_A'=v_A(t)} \frac{dv_A'}{u - v_A'} = \int_{t'=0}^{t'=t} \frac{dt'}{m_{A,0} + bt'} \tag{12.1.37}$$

yielding

$$\ln\frac{u}{u - v_A(t)} = \ln\frac{m_{A,0} + bt}{m_{A,0}}. \tag{12.1.38}$$

Again exponentiating both sides yields

$$\frac{u}{u - v_A(t)} = \frac{m_{A,0} + bt}{m_{A,0}} . \tag{12.1.39}$$

and after some algebraic manipulation we can find the speed of the cart as a function of time

$$v_A(t) = \frac{bt}{m_{A,0} + bt} u . \tag{12.1.40}$$

in agreement with Eq. (12.1.34).

12.3 Rocket Propulsion

A rocket at time $t = 0$ is moving with speed $v_{r,0}$ in the positive x-direction. The rocket burns fuel that is then ejected backward with velocity $\vec{\mathbf{u}} = -u\hat{\mathbf{i}}$ relative to the rocket, where $u > 0$ is the relative speed of the ejected fuel. This exhaust velocity is independent of the velocity of the rocket. The rocket must exert a force to accelerate the ejected fuel backwards and therefore by Newton's Third law, the fuel exerts a force that is equal in magnitude but opposite in direction resulting in propelling the rocket forward. The rocket velocity is a function of time, $\vec{\mathbf{v}}_r(t) = v_{r,x}(t)\hat{\mathbf{i}}$, and the x-component increases at a rate $dv_{r,x} / dt$. Because fuel is leaving the rocket, the mass of the rocket is also a function of time, $m_r(t)$, and is decreasing at a rate dm_r / dt. We shall use the momentum principle, Eq. (12.1.2), to determine a differential equation that relates $dv_{r,x} / dt$, dm_r / dt, u, $v_{r,x}(t)$, and $F_{ext,x}$, an equation known as the rocket equation.

Let $t = t_i$ denote the instant the rocket begins to burn fuel and let $t = t_f$ denote the instant the rocket has finished burning fuel. At some arbitrary time t during this process, the rocket has velocity $\vec{\mathbf{v}}_r(t) = v_{r,x}(t)\hat{\mathbf{i}}$, with the mass of the rocket denoted by $m_r(t) \equiv m_r$. During the time interval $[t, t + \Delta t]$, with Δt taken to be a small interval (we shall eventually consider the limit that $\Delta t \to 0$), a small amount of fuel of mass Δm_f (in the limit that $\Delta t \to 0$, $\Delta m_f \to 0$) is ejected backwards with speed u relative to the rocket. The fuel was initially traveling at the speed of the rocket and so undergoes a change in momentum. The rocket recoils forward, undergoing a change in momentum. In order to keep track of all momentum changes, we define our system to be the rocket (including all the fuel that is not burned during the time interval Δt) and the small amount of fuel that is ejected during the interval Δt. At time t, the fuel has not yet been ejected so it is still inside the rocket. Figure 12.14 represents a momentum diagram at time t for our system relative to a fixed inertial reference frame in which the rocket at time t is moving with speed $v_{r,x}(t)$.

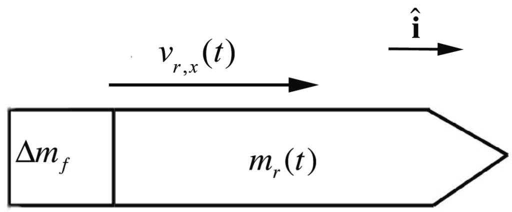


Figure 12.14 Momentum diagram for system at time t

The x-component of the momentum of the system at time t is therefore

$$p_{sys,x}(t) = (m_r(t) + \Delta m_f) v_{r,x}(t). \tag{12.1.41}$$

During the interval $[t, t + \Delta t]$ the fuel is ejected backwards relative to the rocket with speed u. The rocket recoils forward with an increased x-component of the velocity $v_{r,x}(t + \Delta t) = v_{r,x}(t) + \Delta v_{r,x}$, where $\Delta v_{r,x}$ represents the increase the rocket's x-component of the velocity. As usual let's assume that the fuel element, with mass Δm_f, has left the rocket at the end of the time interval, so that the x-component of the velocity of the fuel is $v_{f,x} = v_{r,x} + \Delta v_{r,x} - u$. The momentum diagram of the system at time $t + \Delta t$ is shown in Figure 12.15.

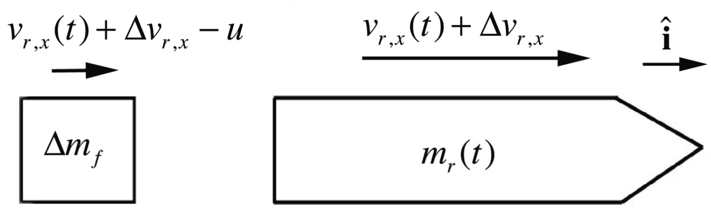


Figure 12.15 Momentum diagram for system at time $t + \Delta t$

The x-component of the momentum of the system at time $t + \Delta t$ is therefore

$$p_{sys,x}(t + \Delta t) = m_r(t)(v_{r,x}(t) + \Delta v_{r,x}) + \Delta m_f (v_{r,x}(t) + \Delta v_{r,x} - u). \tag{12.1.42}$$

In Figure 12.16, we show the diagram depicting the change in the x-component of the momentum of the system consisting of the ejected fuel and rocket.

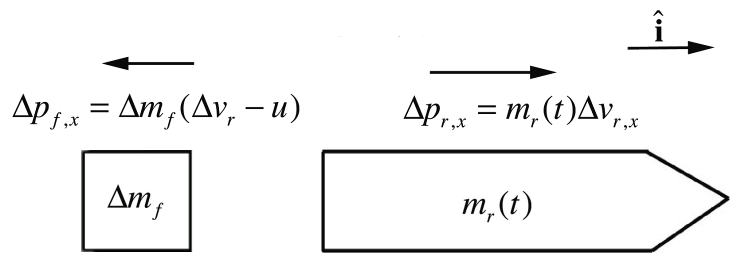


Figure 12.16 Change in momentum for system during time interval $[t, t+\Delta t]$

Therefore the change in the x-component of the momentum of the system is given by

$$\Delta p_{sys,x} = \Delta p_{r,x} + \Delta p_{f,x} = m_r(t)\Delta v_{r,x} + \Delta m_f(\Delta v_{r,x} - u)\,. \qquad (12.1.43)$$

We again note that $\Delta p_{f,x} = \Delta m_f(\Delta v_{r,x} - u) \simeq -\Delta m_f u$, and we show the modified diagram for the change in the x-component of the momentum of the system in Figure 12.17.

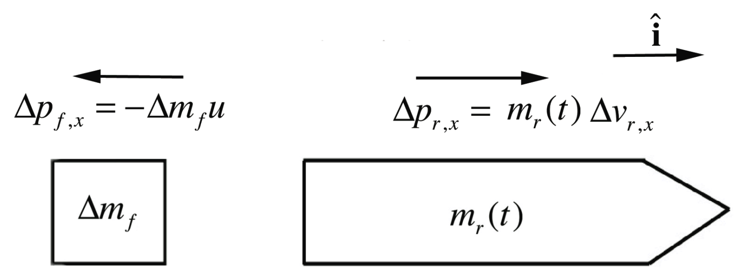


Figure 12.17 Modified change in momentum for system during time interval $[t, t+\Delta t]$

We can now apply Newton's Second Law in the form of the momentum principle (Eq. (12.1.2)), for the system consisting of the rocket and exhaust fuel,

$$F_{ext,x} = \lim_{\Delta t \to 0} \frac{p_{sys,x}(t+\Delta t) - p_{sys,x}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p_{sys,x}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p_{r,x}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta p_{f,x}}{\Delta t}\,. \qquad (12.1.44)$$

From our diagram depicting the change in the x-component of the momentum of the system, we have that

$$F_{ext,x} = \lim_{\Delta t \to 0} \frac{m_r(t)\Delta v_{r,x}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_f(\Delta v_{r,x} - u)}{\Delta t}\,. \qquad (12.1.45)$$

We note that $\Delta m_f \Delta v_{r,x}$ is a second order differential, therefore

$$\lim_{\Delta t \to 0} \frac{\Delta m_f \Delta v_{r,x}}{\Delta t} = 0\,. \qquad (12.1.46)$$

We also note that

$$\frac{dv_{r,x}}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta v_{r,x}}{\Delta t}, \qquad (12.1.47)$$

and

$$\frac{dm_f}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta m_f}{\Delta t}. \qquad (12.1.48)$$

Therefore Eq. (12.1.45) becomes

$$F_{ext,x} = m_r(t)\frac{dv_{r,x}}{dt} - \frac{dm_f}{dt}u. \qquad (12.1.49)$$

We can rewrite Eq. (12.1.49) as

$$F_{ext,x} + \frac{dm_f}{dt}u = m_r(t)\frac{dv_{r,x}}{dt}. \qquad (12.1.50)$$

The second term on the left-hand-side of Eq. (12.1.50) is called the ***thrust***

$$F_{thrust,x} \equiv \frac{dm_f}{dt}u. \qquad (12.1.51)$$

Note that this is not an extra force but the result of the forward recoil due to the ejection of the fuel. Because we are burning fuel at a positive rate $dm_f / dt > 0$ and the speed $u > 0$, the direction of the thrust is in the positive x-direction.

The rate of decrease of the mass of the rocket, dm_r / dt, is equal to the negative of the rate of increase of the exhaust fuel

$$\frac{dm_r}{dt} = -\frac{dm_f}{dt}. \qquad (12.1.52)$$

Therefore substituting Eq. (12.1.52) into Eq. (12.1.49), we find that the differential equation describing the motion of the rocket and exhaust fuel is given by

$$F_{ext,x} - \frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt}. \qquad (12.1.53)$$

Eq. (12.1.53) is called the ***rocket equation***.

12.3.1 Rocket Equation in Gravity-free Space

We shall first consider the case in which there are no external forces acting on the system, then Eq. (12.1.53) becomes

$$-\frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt}\,. \qquad (12.1.54)$$

In order to solve this equation, we separate the variable quantities $v_{r,x}(t)$ and $m_r(t)$

$$\frac{dv_{r,x}}{dt} = -\frac{u}{m_r(t)}\frac{dm_r}{dt}\,. \qquad (12.1.55)$$

We now multiply both sides by dt and integrate with respect to time between the initial time t_i when the ejection of the burned fuel began and the final time t_f when the process stopped.

$$\int_{t'=t_i}^{t'=t_f} \frac{dv_{r,x}}{dt'}dt' = -\int_{t'=t_i}^{t'=t_f} \frac{u}{m_r(t)}\frac{dm_r}{dt'}dt'\,. \qquad (12.1.56)$$

We can rewrite the integrands and endpoints as

$$\int_{v'_{r,x}=v_{r,x,i}}^{v'_{r,x}=v_{r,x,f}} dv'_{r,x} = -\int_{m'_r=m_{r,i}}^{m'_r=m_{r,f}} \frac{u}{m'_r}dm'_r\,. \qquad (12.1.57)$$

Performing the integration and substituting in the values at the endpoints gives

$$v_{r,x,f} - v_{r,x,i} = -u\ln\left(\frac{m_{r,f}}{m_{r,i}}\right). \qquad (12.1.58)$$

Because the rocket is losing fuel, $m_{r,f} < m_{r,i}$, we can rewrite Eq. (12.1.58) as

$$v_{r,x,f} - v_{r,x,i} = u\ln\left(\frac{m_{r,i}}{m_{r,f}}\right). \qquad (12.1.59)$$

We note $\ln(m_{r,i} / m_{r,f}) > 1$. Therefore $v_{r,x,f} > v_{r,x,i}$, as we expect.

After a slight rearrangement of Eq. (12.1.59), we have an expression for the velocity of the rocket as a function of the mass m_r of the rocket

$$v_{r,x,f} = v_{r,x,i} + u\ln\left(\frac{m_{r,i}}{m_{r,f}}\right). \qquad (12.1.60)$$

Let's examine our result. First, let's suppose that all the fuel was burned and ejected. Then $m_{r,f} \equiv m_{r,d}$ is the final dry mass of the rocket (empty of fuel). The ratio

$$R = \frac{m_{r,i}}{m_{r,d}} \tag{12.1.61}$$

is the ratio of the initial mass of the rocket (including the mass of the fuel) to the final dry mass of the rocket (empty of fuel). The final velocity of the rocket is then

$$v_{r,x,f} = v_{r,x,i} + u\ln R\,. \tag{12.1.62}$$

This is why multistage rockets are used. You need a big container to store the fuel. Once all the fuel is burned in the first stage, the stage is disconnected from the rocket. During the next stage the dry mass of the rocket is much less and so R is larger than the single stage, so the next burn stage will produce a larger final speed then if the same amount of fuel were burned with just one stage (more dry mass of the rocket). In general rockets do not burn fuel at a constant rate but if we assume that the burning rate is constant where

$$b = \frac{dm_f}{dt} = -\frac{dm_r}{dt} \tag{12.1.63}$$

then we can integrate Eq. (12.1.63)

$$\int_{m_r' = m_{r,i}}^{m_r' = m_r(t)} dm_r' = -b\int_{t'=t_i}^{t'=t} dt' \tag{12.1.64}$$

and find an equation that describes how the mass of the rocket changes in time

$$m_r(t) = m_{r,i} - b(t - t_i)\,. \tag{12.1.65}$$

For this special case, if we set $t_f = t$ in Eq. (12.1.60), then the velocity of the rocket as a function of time is given by

$$v_{r,x,f} = v_{r,x,i} + u\ln\left(\frac{m_{r,i}}{m_{r,i} - bt}\right). \tag{12.1.66}$$

Example 12.4 Single-Stage Rocket

Before a rocket begins to burn fuel, the rocket has a mass of $m_{r,i} = 2.81\times 10^7\,\text{kg}$, of which the mass of the fuel is $m_{f,i} = 2.46\times 10^7\,\text{kg}$. The fuel is burned at a constant rate with total burn time is $510\,\text{s}$ and ejected at a speed $u = 3000\,\text{m/s}$ relative to the rocket. If the rocket starts from rest in empty space, what is the final speed of the rocket after all the fuel has been burned?

Solution: The dry mass of the rocket is $m_{r,d} \equiv m_{r,i} - m_{f,i} = 0.35\times 10^{7}\,\text{kg}$, hence $R = m_{r,i} / m_{r,d} = 8.03$. The final speed of the rocket after all the fuel has burned is

$$v_{r,f} = \Delta v_r = u\ln R = 6250\ \text{m/s}\ . \qquad (12.1.67)$$

Example 12.5 Two-Stage Rocket

Now suppose that the same rocket in Example 12.4 burns the fuel in two stages ejecting the fuel in each stage at the same relative speed. In stage one, the available fuel to burn is $m_{f,1,i} = 2.03\times 10^{7}\,\text{kg}$ with burn time $150\ \text{s}$. Then the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket. These disconnected parts have a mass $m = 1.4\times 10^{6}\,\text{kg}$. All the remaining fuel with mass is burned during the second stage with burn time of $360\ \text{s}$. What is the final speed of the rocket after all the fuel has been burned?

Solution: The mass of the rocket after all the fuel in the first stage is burned is $m_{r,1,d} = m_{r,1,i} - m_{f,1,i} = 0.78\times 10^{7}\,\text{kg}$ and $R_1 = m_{r,1,i} / m_{r,1,d} = 3.60$. The change in speed after the first stage is complete is

$$\Delta v_{r,1} = u\ln R_1 = 3840\ \text{m/s}. \qquad (12.1.68)$$

After the empty fuel tank and accessories from stage one are disconnected from the rest of the rocket, the remaining mass of the rocket is $m_{r,2,d} = 2.1\times 10^{6}\,\text{kg}$. The remaining fuel has mass $m_{f,2,i} = 4.3\times 10^{6}\,\text{kg}$. The mass of the rocket plus the unburned fuel at the beginning of the second stage is $m_{r,2,i} = 6.4\times 10^{6}\,kg$. Then $R_2 = m_{r,2,i} / m_{r,2,d} = 3.05$. Therefore the rocket increases its speed during the second stage by an amount

$$\Delta v_{r,2} = u\ln R_2 = 3340\ \text{m/s}. \qquad (12.1.69)$$

The final speed of the rocket is the sum of the change in speeds due to each stage,

$$v_f = \Delta v_r = u\ln R_1 + u\ln R_2 = u\ln(R_1 R_2) = 7190\ \text{m/s}, \qquad (12.1.70)$$

which is greater than if the fuel were burned in one stage. Plots of the speed of the rocket as a function time for both one-stage and two-stage burns are shown Figure 12.18.

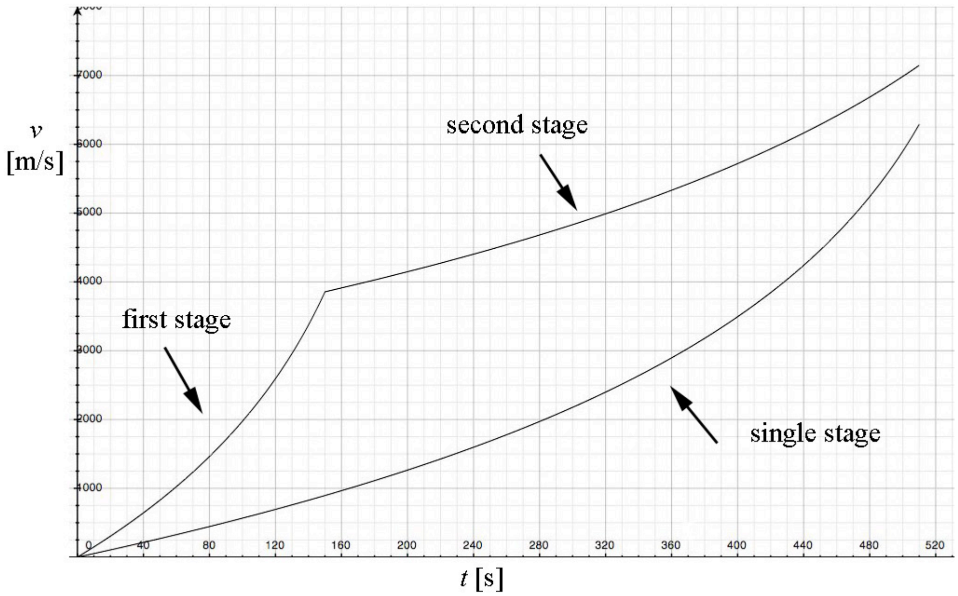


Figure 12.18 Plots of speed of rocket for both one-stage burn and two-stage burn

12.3.2 Rocket in a Constant Gravitational Field:

Now suppose that the rocket takes off from rest at time $t = 0$ in a constant gravitational field then the external force is

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = m_r \vec{\mathbf{g}} . \tag{12.1.71}$$

Choose the positive x-axis in the upward direction then $F_{ext,x}(t) = -m_r(t)g$. Then the rocket equation (Eq. (12.1.53) becomes

$$-m_r(t)g - \frac{dm_r}{dt}u = m_r(t)\frac{dv_{r,x}}{dt} . \tag{12.1.72}$$

Multiply both sides of Eq. (12.1.72) by dt, and divide both sides by $m_r(t)$. Then Eq. (12.1.72) can be written as

$$dv_{r,x} = -gdt - \frac{dm_r}{m_r(t)}u . \tag{12.1.73}$$

We now integrate both sides

$$\int_{v_{r,x,i}=0}^{v_{r,x}(t)} dv'_{r,x} = -u \int_{m_{r,i}}^{m_r(t)} \frac{dm'_r}{m'_r} - g\int_0^t dt' , \tag{12.1.74}$$

where $m_{r,i}$ is the initial mass of the rocket and the fuel. Integration yields

$$v_{r,x}(t) = -u\ln\left(\frac{m_r(t)}{m_{r,i}}\right) - gt = u\ln\left(\frac{m_{r,i}}{m_r(t)}\right) - gt\ . \qquad (12.1.75)$$

After all the fuel is burned at $t = t_f$, the mass of the rocket is equal to the dry mass $m_{r,f} = m_{r,d}$ and so

$$v_{r,x}(t_f) = u\ln R - gt_f\ . \qquad (12.1.76)$$

The first term on the right hand side is independent of the burn time. However the second term depends on the burn time. The shorter the burn time, the smaller the negative contribution from the third turn, and hence the larger the final speed. So the rocket engine should burn the fuel as fast as possible in order to obtain the maximum possible speed.