Chapter 27 Static Fluids

Chapter 27 Static Fluids

27.1 Introduction

Water is everywhere around us, covering 71% of the Earth's surface. The water content of a human being can vary between 45% and 70 % of body weight. Water can exist in three states of matter: solid (ice), liquid, or gas. Water flows through many objects: through rivers, streams, aquifers, irrigation channels, and pipes to mention a few. Humans have tried to control and harness this flow through many different technologies such as aqueducts, Archimedes' screw, pumps, and water turbines. Water in the gaseous state also flows. Water vapor, lighter than air, can cause convection currents that form clouds. In the liquid state, the density of water molecules is greater than the gaseous state but in both states water can flow. Liquid water forms a surface while water vapor does not. Water in both the liquid and gaseous state is classified as a *fluid* to distinguish it from the solid state. There is some ambiguity in the use of the term fluid. Ice flows in a glacier but very slowly. So for a short time interval compared to the time interval involved in the flow, glacial ice can be thought of as a solid. In ordinary language, the term fluid is used to describe the liquid state of matter but a fluid is any state of matter that flows when there is an applied shear stress. The *viscosity* of a fluid is a measure of its resistance to gradual deformation by shear stress or tensile stress.

27.2 Density

The density of a small amount of matter is defined to be the amount of mass ΔM divided by the volume ΔV of that element of matter,

$$\rho = \Delta M / \Delta V \quad . \qquad (27.2.1)$$

The SI unit for density is the kilogram per cubic meter, $\text{kg} \cdot \text{m}^{-3}$. If the density of a material is the same at all points, then the density is given by

$$\rho = M / V \quad , \qquad (27.2.2)$$

where M is the mass of the material and V is the volume of material. A material with constant density is called *homogeneous.* For a homogeneous material, density is an *intrinsic* property. If we divide the material in two parts, the density is the same in both parts,

$$\rho = \rho_1 = \rho_2 \quad . \qquad (27.2.3)$$

However mass and volume are *extrinsic* properties of the material. If we divide the material into two parts, the mass is the sum of the individual masses

$$M = M_1 + M_2 \quad , \qquad (27.2.4)$$

as is the volume

$$V = V_1 + V_2 \ . \tag{27.2.5}$$

The density is tabulated for various materials in Table 27.1.

Table 27.1: Density for Various Materials (Unless otherwise noted, all densities given are at standard conditions for temperature and pressure, that is, 273.15 K (0.00 °C) and 100 kPa (0.987 atm).

Material	Density, ρ $\text{kg}\cdot\text{m}^{-3}$
Helium	0.179
Air (at sea level)	1.20
Styrofoam	75
Wood Seasoned, typical	0.7×10^3
Ethanol	0.81×10^3
Ice	0.92×10^3
Water	1.00×10^3
Seawater	1.03×10^3
Blood	1.06×10^3
Aluminum	2.70×10^3
Iron	7.87×10^3
Copper	8.94×10^3
Lead	11.34×10^3
Mercury	13.55×10^3
Gold	19.32×10^3
Plutonium	19.84×10^3
Osmium	22.57×10^3

27.3 Pressure in a Fluid

When a shear force is applied to the surface of fluid, the fluid will undergo flow. When a fluid is static, the force on any surface within fluid must be perpendicular (normal) to each side of that surface. This force is due to the collisions between the molecules of the fluid on one side of the surface with molecules on the other side. For a static fluid, these forces must sum to zero. Consider a small portion of a static fluid shown in Figure 27.1.

That portion of the fluid is divided into two parts, which we shall designate 1 and 2, by a small mathematical shared surface element S of area A_S. The force $\vec{\mathbf{F}}_{1,2}(S)$ on the surface of region 2 due to the collisions between the molecules of 1 and 2 is perpendicular to the surface.

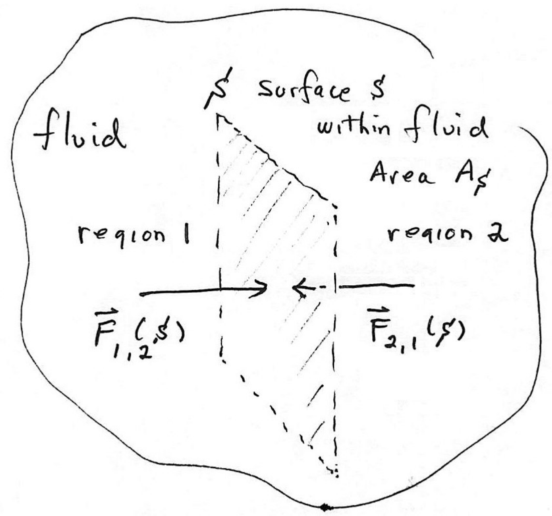

Figure 27.1: Forces on a surface within a fluid

The force $\vec{\mathbf{F}}_{2,1}(S)$ on the surface of region 1 due to the collisions between the molecules of 1 and 2 by Newton's Third Law satisfies

$$\vec{\mathbf{F}}_{1,2}(S) = -\vec{\mathbf{F}}_{2,1}(S) \; . \qquad (27.3.1)$$

Denote the magnitude of these forces that form this interaction pair by

$$F_{\perp}(S) = \left|\vec{\mathbf{F}}_{1,2}(S)\right| = \left|\vec{\mathbf{F}}_{2,1}(S)\right| \; . \qquad (27.3.2)$$

Define the **hydrostatic pressure** at those points within the fluid that lie on the surface S by

$$P \equiv \frac{F_{\perp}(S)}{A_S} \; . \qquad (27.3.3)$$

The pressure at a point on the surface S is the limit

$$P = \lim_{A_S \to 0} \frac{F_{\perp}(S)}{A_S} \; . \qquad (27.3.4)$$

The SI units for pressure are $\mathrm{N \cdot m^{-2}}$ and is called the **pascal** (Pa), where

$$1\,\text{Pa} = 1\,\text{N}\cdot\text{m}^{-2} = 10^{-5}\ \text{bar}\ . \qquad (27.3.5)$$

Atmospheric pressure at a point is the force per unit area exerted on a small surface containing that point by the weight of air above that surface. In most circumstances atmospheric pressure is closely approximated by the hydrostatic pressure caused by the weight of air above the measurement point. On a given surface area, low-pressure areas have less atmospheric mass above their location, whereas high-pressure areas have more atmospheric mass above their location. Likewise, as elevation increases, there is less overlying atmospheric mass, so that atmospheric pressure decreases with increasing elevation. On average, a column of air one square centimeter in cross-section, measured from sea level to the top of the atmosphere, has a mass of about 1.03 kg and weight of about 10.1 N. (A column one square inch in cross-section would have a weight of about 14.7 lbs, or about 65.4 N). The **standard atmosphere** [atm] is a unit of pressure such that

$$1\,\text{atm} = 1.01325\times10^{5}\ \text{Pa} = 1.01325\ \text{bar}\ . \qquad (27.3.6)$$

27.4 Pascal's Law: Pressure as a Function of Depth in a Fluid of Uniform Density in a Uniform Gravitational Field

Consider a static fluid of uniform density ρ. Choose a coordinate system such that the z-axis points vertical downward and the plane $z = 0$ is at the surface of the fluid. Choose an infinitesimal cylindrical volume element of the fluid at a depth z, cross-sectional area A and thickness dz as shown in Figure 27.3. The volume of the element is $dV = A\,dz$ and the mass of the fluid contained within the element is $dm = \rho A\,dz$.

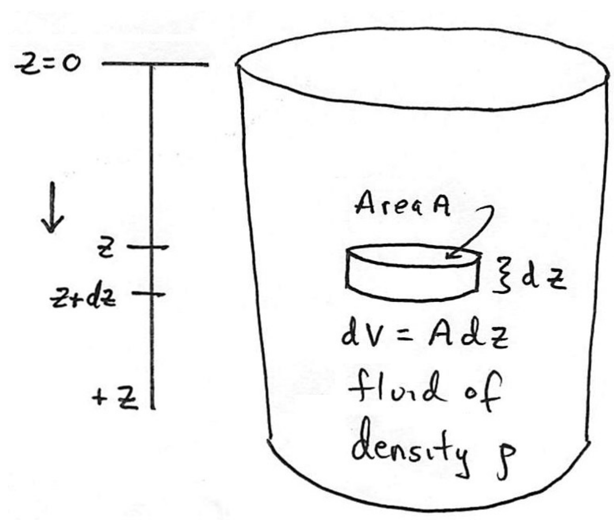

Figure 27.2: Coordinate system for fluid

The surface of the infinitesimal fluid cylindrical element has three faces, two caps and the cylindrical body. Because the fluid is static the force due to the fluid pressure points inward on each of these three faces. The forces on the cylindrical surface add to zero. On the end-cap at z, the force due to pressure of the fluid above the end-cap is downward, $\vec{\mathbf{F}}(z) = F(z)\hat{\mathbf{k}}$, where $F(z)$ is the magnitude of the force. On the end-cap at $z+dz$, the force due to the pressure of the fluid below the end-cap is upward, $\vec{\mathbf{F}}(z+dz) = -F(z+dz)\hat{\mathbf{k}}$, where $F(z+dz)$ is the magnitude of the force. The gravitational force acting on the element is given by $\vec{\mathbf{F}}^{g} = (dm)g\hat{\mathbf{k}} = (\rho dV)g\hat{\mathbf{k}} = \rho A\, dz\, g\hat{\mathbf{k}}$. There are also radial inward forces on the cylindrical body which sum to zero. The free body force diagram on the element is shown in Figure 27.3.

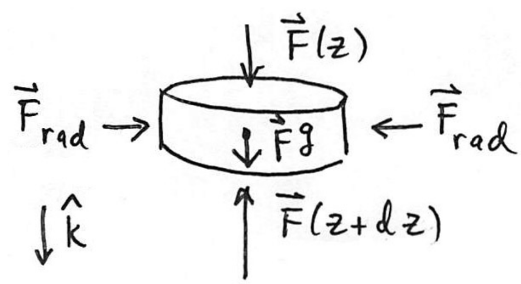

Figure 27.3: Free-body force diagram on cylindrical fluid element

The vector sum of the forces is zero because the fluid is static (Newton's Second Law). Therefore in the $+\hat{\mathbf{k}}$-direction

$$F(z) - F(z+dz) + \rho A dz g = 0 \ . \tag{27.4.1}$$

We divide through by the area A of the end-cap and use Eq. (27.3.4) to rewrite Eq. (27.4.1) in terms of the pressure

$$P(z) - P(z+dz) + \rho dz g = 0 \ . \tag{27.4.2}$$

Rearrange Eq. (27.4.2) as

$$\frac{P(z+dz) - P(z)}{dz} = \rho g \ . \tag{27.4.3}$$

Now take the limit of Eq. (27.4.3) as the thickness of the element $dz \to 0$,

$$\lim_{dz \to 0} \frac{P(z+dz) - P(z)}{dz} = \rho g \ . \tag{27.4.4}$$

resulting in the differential equation

$$\frac{dP}{dz} = \rho g \, . \tag{27.4.5}$$

Integrate Eq. (27.4.5),

$$\int_{P(z=0)}^{P(z)} dP = \int_{z'=0}^{z'=z} \rho g \, dz' . \qquad (27.4.6)$$

Performing the integrals on both sides of Eq. (27.4.6) describes the change in pressure between a depth z and the surface of a fluid

$$P(z) - P(z=0) = \rho g z \qquad \text{(Pascal's Law)}, \qquad (27.4.7)$$

a result known as **Pascal's Law**.

Example 27.1 Pressure in the Earth's Ocean

What is the change in pressure between a depth of $4\,\text{km}$ and the surface in Earth's ocean?

Solution: We begin by assuming the density of water is uniform in the ocean, and so we can use Pascal's Law, Eq. (27.4.7) to determine the pressure, where we use $\rho = 1.03 \times 10^3\ \text{kg} \cdot \text{m}^{-3}$ for the density of seawater (Table 27.1). Then

$$\begin{aligned} &P(z) - P(z=0) = \rho g z \\ &= (1.03 \times 10^3 \text{kg} \cdot \text{m}^{-3})(9.8\ \text{m} \cdot \text{s}^{-2})(4 \times 10^3\ \text{m}) \\ &= 40 \times 10^6\ \text{Pa}. \end{aligned} \qquad (27.4.8)$$

Example 27.2 Pressure in a Rotating Sample in a Centrifuge

In an ultra centrifuge, a liquid filled chamber is spun with a high angular speed ω about a fixed axis. The density ρ of the fluid is uniform. The open-ended side of the chamber is a distance r_0 from the fixed axis. The chamber has cross sectional area A and of length L, (Figure 27.4).

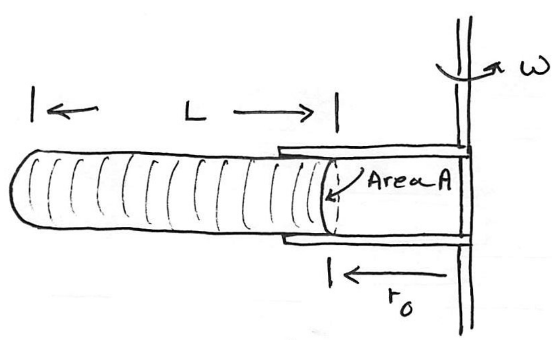

Figure 27.4: Schematic representation of centrifuge

The chamber is spinning fast enough to ignore the effect of gravity. Determine the pressure in the fluid as a function of distance r from the fixed axis.

Solution: Choose polar coordinates in the plane of circular motion. Consider a small volume element of the fluid of cross-sectional area A, thickness dr, and mass $dM = \rho A dr$ that is located a distance r from the fixed axis. Denote the pressure at one end of the volume element by $P(r) = F(r)/A$ and the pressure at the other end by $P(r+dr) = F(r+dr)/A$. The free-body force diagram on the volume fluid element is shown in Figure 27.5.

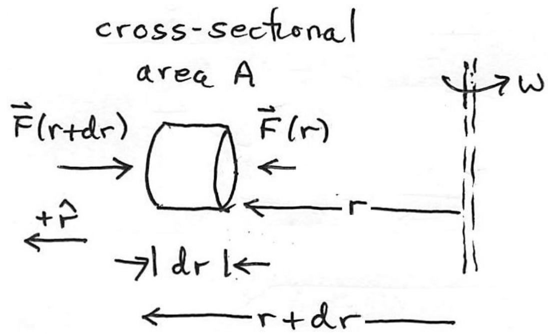

Figure 27.5: Free-body force diagram showing only radial forces on fluid element in centrifuge

The element is accelerating inward with radial component of the acceleration, $a_r = -r\omega^2$. Newton's Second Law applied to the fluid element is then

$$(P(r) - P(r+dr))A = -(\rho A dr) r\omega^2, \tag{27.4.9}$$

We can rewrite Eq. (27.4.9) as

$$\frac{P(r+dr) - P(r)}{dr} = \rho r\omega^2, \tag{27.4.10}$$

and take the limit $dr \to 0$ resulting in

$$\frac{dP}{dr} = \rho r\omega^2. \tag{27.4.11}$$

We can integrate Eq. (27.4.11) between an arbitrary distance r from the rotation axis and the open-end located at r_0, where the pressure $P(r_0) = 1\,\text{atm}$,

$$\int_{P(r)}^{P(r_0)} dP = \rho\omega^2 \int_{r'=r}^{r'=r_0} r'\,dr'. \tag{27.4.12}$$

Integration yields

$$P(r_0) - P(r) = \frac{1}{2}\rho\omega^2(r_0^{\ 2} - r^2) . \qquad (27.4.13)$$

The pressure at a distance r from the rotation axis is then

$$P(r) = P(r_0) + \frac{1}{2}\rho\omega^2(r^2 - r_0^{\ 2}) . \qquad (27.4.14)$$

27.5 Compressibility of a Fluid

When the pressure is uniform on all sides of an object in a fluid, the pressure will squeeze the object resulting in a smaller volume. When we increase the pressure by ΔP on a material of volume V_0, then the volume of the material will change by $\Delta V < 0$ and consequently the density of the material will also change. Define the *bulk stress* by the increase in pressure change

$$\sigma_B \equiv \Delta P \ . \qquad (27.5.1)$$

Define the *bulk strain* by the ratio

$$\varepsilon_B \equiv \frac{\Delta V}{V_0} \ . \qquad (27.5.2)$$

For many materials, for small pressure changes, the bulk stress is linearly proportional to the bulk strain,

$$\Delta P = -B\frac{\Delta V}{V_0} \ , \qquad (27.5.3)$$

where the constant of proportionality B is called the **bulk modulus**. The SI unit for bulk modulus is the pascal. If the bulk modulus of a material is very large, a large pressure change will result in only a small volume change. In that case the material is called *incompressible*. In Table 27.2, the bulk modulus is tabulated for various materials.

Table 27.2 Bulk Modulus for Various Materials

Material	Bulk Modulus, Y, (Pa)
Diamond	4.4×10^{11}
Iron	1.6×10^{11}
Nickel	1.7×10^{11}
Steel	1.6×10^{11}
Copper	1.4×10^{11}
Brass	6.0×10^{10}
Aluminum	7.5×10^{10}
Crown Glass	5.0×10^{10}
Lead	4.1×10^{10}
Water (value increases at higher pressure)	2.2×10^{9}
Air (adiabatic bulk modulus)	1.42×10^{5}
Air (isothermal bulk modulus)	1.01×10^{5}

Example 27.3 Compressibility of Water

Determine the percentage decrease in a fixed volume of water at a depth of 4 km where the pressure difference is 40 MPa, with respect to sea level.

Solution: The bulk modulus of water is $2.2\times10^9\text{ Pa}$. From Eq. (27.5.3),

$$\frac{\Delta V}{V_0}=-\frac{\Delta P}{B}=-\frac{40\times10^6\,\text{Pa}}{2.2\times10^9\text{ Pa}}=-0.018\,; \qquad (27.5.4)$$

there is only a 1.8% decrease in volume. Water is essentially incompressible even at great depths in ocean, justifying our assumption that the density of water is uniform in the ocean in Example 27.1.

27.6 Archimedes' Principle: Buoyant Force

When we place a piece of solid wood in water, the wood floats on the surface. The density of most woods is less than the density of water, and so the fact that wood floats does not seem so surprising. However, objects like ships constructed from materials like steel that are much denser than water also float. In both cases, when the floating object is at rest, there must be some other force that exactly balances the gravitational force. This balancing of forces also holds true for the fluid itself.

Consider a static fluid with uniform density ρ_f. Consider an arbitrary volume element of the fluid with volume V and mass $m_f = \rho_f V$. The gravitational force acts on the volume element, pointing downwards, and is given by $\vec{\mathbf{F}}^g = -\rho_f V g \hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is a unit vector pointing in the upward direction. The pressure on the surface is perpendicular to the surface (Figure 27.6). Therefore on each area element of the surface there is a perpendicular force on the surface.

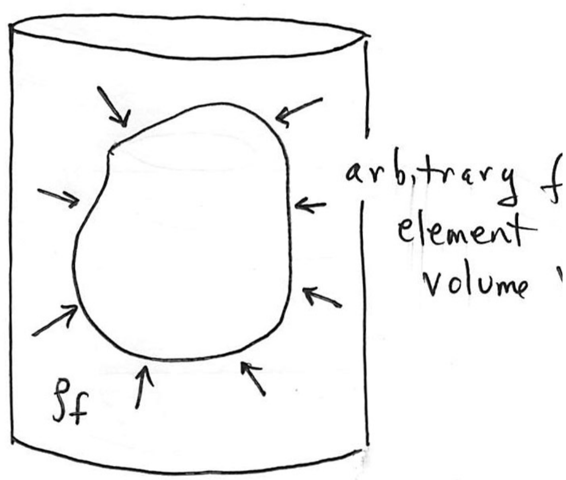

Figure 27.6: Forces due to pressure on surface of arbitrary volume fluid element

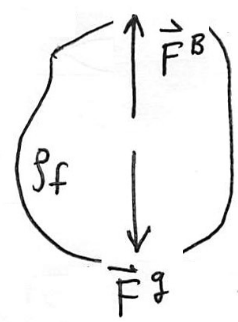

Figure 27.7: Free-body force diagram on volume element showing gravitational force and buoyant force

Let $\vec{\mathbf{F}}^B$ denote the resultant force, called the **buoyant force**, on the surface of the volume element due to the pressure of the fluid. The buoyant force must exactly balance the gravitational force because the fluid is in static equilibrium (Figure 27.7),

$$\vec{\mathbf{0}} = \vec{\mathbf{F}}^B + \vec{\mathbf{F}}^g = \vec{\mathbf{F}}^B - \rho_f V g \hat{\mathbf{k}} . \tag{27.6.1}$$

Therefore the buoyant force is therefore

$$\vec{\mathbf{F}}^B = \rho_f V g \hat{\mathbf{k}} . \tag{27.6.2}$$

The buoyant force depends on the density of the fluid, the gravitational constant, and the volume of the fluid element. This macroscopic description of the buoyant force that results from a very large number of collisions of the fluid molecules is called **Archimedes' Principle.**

We can now understand why when we place a stone in water it sinks. The density of the stone is greater than the density of the water, and so the buoyant force on the stone is less than the gravitational force on the stone and so it accelerates downward.

Place a uniform object of volume V and mass M with density $\rho_o = M / V$ within a fluid. If the density of the object is less than the density of the fluid, $\rho_o < \rho_f$, the object will float on the surface of the fluid. A portion of the object that is a beneath the surface, displaces a volume V_1 of the fluid. The portion of the object that is above the surface displaces a volume $V_2 = V - V_1$ of air (Figure 27.8).

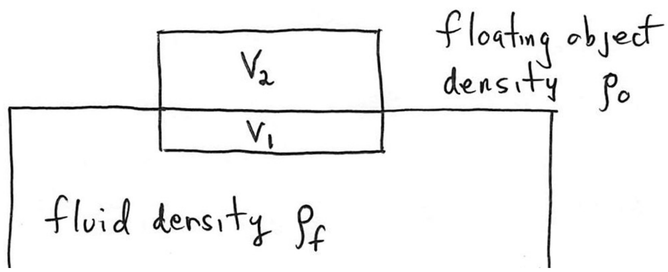

Figure 27.8: Floating object on surface of fluid

Because the density of the air is much less that the density of the fluid, we can neglect the buoyant force of the air on the object.

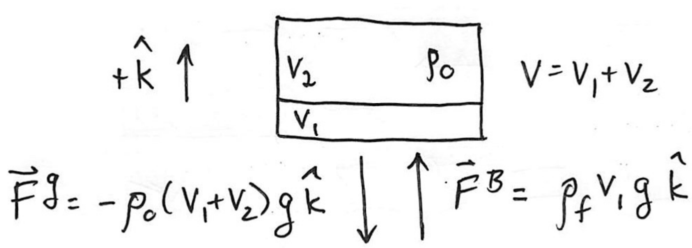

Figure 27.9: Free-body force diagram on floating object

The buoyant force of the fluid on the object, $\vec{\mathbf{F}}^B_{f,o} = \rho_f V_1 g \hat{\mathbf{k}}$, must exactly balance the gravitational force on the object due to the earth, $\vec{\mathbf{F}}^g_{e,o}$ (Figure 27.9),

$$\vec{\mathbf{0}} = \vec{\mathbf{F}}^B_{f,o} + \vec{\mathbf{F}}^g_{e,o} = \rho_f V_1 g \hat{\mathbf{k}} - \rho_o V g \hat{\mathbf{k}} = \rho_f V_1 g \hat{\mathbf{k}} - \rho_o (V_1 + V_2) g \hat{\mathbf{k}} . \qquad (27.6.3)$$

Therefore the ratio of the volume of the exposed and submerged portions of the object must satisfy

$$\rho_f V_1 = \rho_o (V_1 + V_2) . \tag{27.6.4}$$

We can solve Eq. (27.6.4) and determine the ratio of the volume of the exposed and submerged portions of the object

$$\frac{V_2}{V_1} = \frac{(\rho_f - \rho_o)}{\rho_o} . \tag{27.6.5}$$

We now also can understand why a ship of mass M floats. The more dense steel displaces a volume of water V_s but a much larger volume of water V_w is displaced by air. The buoyant force on the ship is then

$$\vec{\mathbf{F}}_s^{\,B} = \rho_f (V_s + V_w) g \,\hat{\mathbf{k}} . \tag{27.6.6}$$

If this force is equal in magnitude to Mg, the ship will float.

Example 27.4 Archimedes' Principle: Floating Wood

Consider a beaker of uniform cross-sectional area A, filled with water of density ρ_w. When a rectangular block of wood of cross sectional area A_2, height, and mass M_b is placed in the beaker, the bottom of the block is at an unknown depth z below the surface of the water. (a) How far below the surface z is the bottom of the block? (b) How much did the height of the water in the beaker rise when the block was placed in the beaker?

Solution: We neglect the buoyant force due to the displaced air because it is negligibly small compared to the buoyant force due to the water. The beaker, with the floating block of wood, is shown in Figure 27.10.

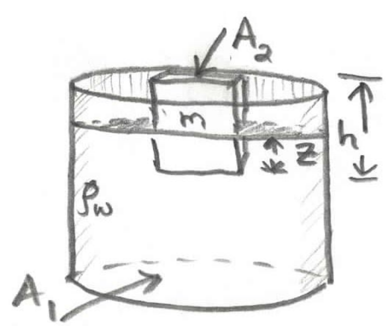

Figure 27.10 Block of wood floating in a beaker of water

(a) The density of the block of wood is $\rho_b = M_b / V_b = M_b / A_b h$. The volume of the submerged portion of the wood is $V_1 = A_b z$. The volume of the block above the surface is given by $V_2 = A_b (h - z)$. We can apply Eq. (27.6.5), and determine that

$$\frac{V_2}{V_1} = \frac{A_b(h-z)}{A_b z} = \frac{(h-z)}{z} = \frac{(\rho_w - \rho_b)}{\rho_b}. \tag{27.6.7}$$

We can now solve Eq. (27.6.7) for the depth z of the bottom of the block

$$z = \frac{\rho_b}{\rho_w}h = \frac{(M_b / A_b h)}{\rho_w}h = \frac{M_b}{\rho_w A_b}. \tag{27.6.8}$$

(b) Before the block was placed in the beaker, the volume of water in the beaker is $V_w = As_i$, where s_i is the initial height of water in the beaker. When the wood is floating in the beaker, the volume of water in the beaker is equal to $V_w = As_f - A_b z$, where s_f is the final height of the water, in the beaker and $A_b z$ is the volume of the submerged portion of block. Because the volume of water has not changed

$$As_i = As_f - A_b z. \tag{27.6.9}$$

We can solve Eq. (27.6.9) for the change in height of the water $\Delta s = s_f - s_i$, in terms of the depth z of the bottom of the block,

$$\Delta s = s_f - s_i = \frac{A_b}{A}z. \tag{27.6.10}$$

We now substitute Eq. (27.6.8) into Eq. (27.6.10) and determine the change in height of the water

$$\Delta s = s_f - s_i = \frac{M_b}{\rho_w A}. \tag{27.6.11}$$

Example 27.5 Rock Inside a Floating Salad Bowl

A rock of mass m_r and density ρ_r is placed in a salad bowl of mass m_b. The salad bowl and rock float in a beaker of water of density ρ_w. The beaker has cross sectional area A. The rock is then removed from the bowl and allowed to sink to the bottom of the beaker. Does the water level rise or fall when the rock is dropped into the water?

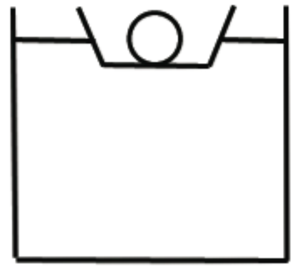

Figure 27.11: Rock in a floating salad bowl

Solution: When the rock is placed in the floating salad bowl, a volume V of water is displaced. The buoyant force $\vec{\mathbf{F}}^B = \rho_w V g \hat{\mathbf{k}}$ balances the gravitational force on the rock and salad bowl,

$$(m_r + m_b)g = \rho_w V g = \rho_w (V_1 + V_2) g \,. \qquad (27.6.12)$$

where V_1 is the portion of the volume of displaced water that is necessary to balance just the gravitational force on the rock, $m_r g = \rho_w V_1 g$, and V_2 is the portion of the volume of displaced water that is necessary to balance just the gravitational force on the bowl, $m_b g = \rho_w V_2 g$, Therefore V_1 must satisfy the condition that $V_1 = m_r g / \rho_w$. The volume of the rock is given by $V_r = m_r / \rho_r$. In particular

$$V_1 = \frac{\rho_r}{\rho_w} V_r \,. \qquad (27.6.13)$$

Because the density of the rock is greater than the density of the water, $\rho_r > \rho_w$, the rock displaces more water when it is floating than when it is immersed in the water, $V_1 > V_r$. Therefore the water level drops when the rock is dropped into the water from the salad bowl.

Example 27.6 Block Floating Between Oil and Water

A cubical block of wood, each side of length $l = 10\text{ cm}$, floats at the interface between air and water. The air is then replaced with $d = 10\text{ cm}$ of oil that floats on top of the water.

a) Will the block rise or fall? Briefly explain your answer.

After the oil has been added and equilibrium established, the cubical block of wood floats at the interface between oil and water with its lower surface $h = 2.0 \times 10^{-2}\text{m}$ below the interface. The density of the oil is $\rho_o = 6.5 \times 10^2 \text{ kg}\cdot\text{m}^{-3}$. The density of water is $\rho_w = 1.0 \times 10^3 \text{ kg}\cdot\text{m}^{-3}$.

b) What is the density of the block of wood?

Solution: (a) The buoyant force is equal to the gravitational force on the block. Therefore

$$\rho_b g V = \rho_w g V_1 + \rho_a g (V - V_1) \ , \tag{27.6.14}$$

where V_1 is the volume of water displaced by the block, $V_2 = V - V_1$ is the volume of air displaced by the block V is the volume of the block, ρ_b is the density of the block of wood, and ρ_a is the density of air (Figure 27.12(a)).

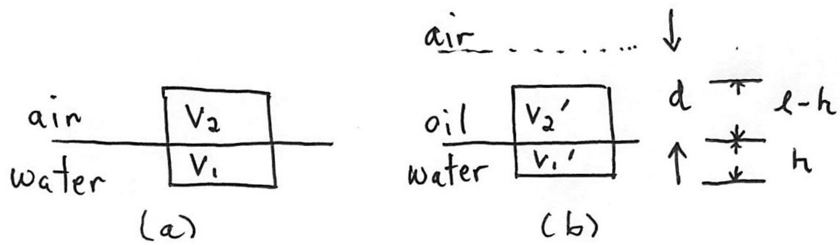

Figure 27.12: (a) Block floating on water, (b) Block floating on oil-water interface

We now solve Eq. (27.6.14) for the volume of water displaced by the block

$$V_1 = \frac{(\rho_b - \rho_a)}{(\rho_w - \rho_a)} V \ . \tag{27.6.15}$$

When the oil is added, we can repeat the argument leading up to Eq. (27.6.15) replacing ρ_a by ρ_o, (Figure 27.12(b)), yielding

$$\rho_b g V = \rho_w g V_1' + \rho_o g V_2' \ , \tag{27.6.16}$$

where V_1' is the volume of water displaced by the block, V_2' is the volume of oil displaced by the block, V is the volume of the block, and ρ_b is the density of the block of wood. Because $V_2' = V - V_1'$, we rewrite Eq. (27.6.16) as

$$\rho_b g V = \rho_w g V_1' + \rho_o g (V - V_1') \ , \tag{27.6.17}$$

We now solve Eq. (27.6.17) for the volume of water displaced by the block,

$$V_1' = \frac{(\rho_b - \rho_o)V}{(\rho_w - \rho_o)} \,. \qquad (27.6.18)$$

Because $\rho_o >> \rho_a$, comparing Eqs. (27.6.18) and (27.6.15), we conclude that $V_1' > V_1$. The block rises when the oil is added because more water is displaced.

(b) We use the fact that $V_1' = l^2 h$, $V_2' = l^2(l - h)$, and $V = l^3$, in Eq. (27.6.16) and solve for the density of the block

$$\rho_b = \frac{\rho_w V_1' + \rho_o V_2'}{V} = \frac{\rho_w l^2 h + \rho_o l^2 (l - h)}{l^3} = (\rho_w - \rho_o)\frac{h}{l} + \rho_o \,. \qquad (27.6.19)$$

We now substitute the given values from the problem statement and find that the density of the block is

$$\begin{aligned} \rho_b &= ((1.0 \times 10^3\,\text{kg}\cdot\text{m}^{-3}) - (6.5 \times 10^2\,\text{kg}\cdot\text{m}^{-3}))\frac{(2.0 \times 10^{-2}\,\text{m})}{(1.0 \times 10^{-1}\,\text{m})} + (6.5 \times 10^2\,\text{kg}\cdot\text{m}^{-3}) \\ \rho_b &= 7.2 \times 10^2\,\text{kg}\cdot\text{m}^{-3}. \end{aligned} \qquad (27.6.20)$$

Because $\rho_b > \rho_o$, the above analysis is valid.