Group Problems #10 - Solutions

Wednesday, September 14

Problem 1 *A Relativistic Electron*

An electron with mass $m_e = 0.511$ MeV/c^2 has a kinetic energy $K = 1.022$ MeV measured in the lab reference frame (S).

(a) What is the total energy E of the electron in frame S? Leave your answer in MeV (mega electron-volts).

The total energy is the sum of kinetic and internal energies:

$$E = m_e c^2 + K = (0.511 + 1.022)\ \text{MeV} = 1.533\ \text{MeV}. \tag{1}$$

Since $E = \gamma m_e c^2$, then we have $\gamma = 3$.

(b) What is the momentum p of the electron in frame S? Leave your answer in MeV/c.

Start with the energy-momentum invariant, $(E/c)^2 - p^2 = m^2c^2$, and solve for p (the magnitude of the 3-momentum):

$$pc = \sqrt{E^2 - m^2c^4} \quad = \quad \sqrt{(1.533)^2 - (0.511)^2} \tag{2}$$

$$\longrightarrow \quad p = 1.445\ \text{MeV/c}. \tag{3}$$

Another way to solve this part is to use the fact that $\gamma = 3$, and then solve for the speed of the electron, u. Finally you can then use the definition of the relativistic momentum of the electron: $p = \gamma m u$. This works, but also takes a little more calculation.

(c) What is the speed of the electron u relative to the speed of light in frame S?

One way to solve this is to take the ratio of the relativistic 3-momentum ($p = \gamma_u m u$) and the total energy ($E = \gamma_u m c^2$):

$$\frac{pc}{E} = \frac{\gamma_u m u c}{\gamma_u m c^2} = \frac{u}{c} = \frac{1.445}{1.533} = 0.943. \tag{4}$$

Another way to solve this is to use $\gamma = 3$ and solve for u.

Now consider the motion of the electron from the point of view of an observer in frame S', which is moving with $v = 0.8c$ relative to S *in a direction opposite to the motion of the electron in* S.

(d) What are the total energy E' and momentum p' of the electron in frame S'?

First calculate the factors needed to perform a Lorentz transformation from the laboratory frame to the S' frame:

$$\beta_v = 4/5, \tag{5}$$

$$\gamma_v = \frac{1}{\sqrt{1-\beta_v^2}} = \frac{1}{\sqrt{1-0.64}} = \frac{5}{3}, \tag{6}$$

$$\beta_v\gamma_v = \frac{4}{5}\cdot\frac{5}{3} = \frac{4}{3}. \tag{7}$$

Note that the subscripts on β and γ are a reminder that they pertain to the relative velocity between the two reference frames, and not to the velocity of the electron in any particular frame. With this understanding, and for clarity, we will drop the subscripts in the calculations below.

To do the Lorentz transformation properly, we must realize that the electron and the frame S' move in opposite directions, as stated in the problem. We must properly account for this in one of two ways: 1) set the direction of motion for S' to be in the $+x$ direction, in which case the electron moves in the $-x$ direction; or 2) set the direction of motion for the S' frame to be in the $-x$ direction, in which case the electron moves in the $+x$ direction. In case 1) you use the familiar form for the Lorentz transformation matrix (shown below), but then must use a negative value for p, the electron's momentum in the S frame. In case 2) we use a positive value for p, but then must use a negative value for β in the Lorentz transformation matrix. I will choose option 1) so $p_x = -p$, and $p_y = p_z = 0$. So we have (in MeV/c):

$$\begin{pmatrix} E'/c \\ p' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E/c \\ -p \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E(5/3) + pc(4/3) \\ -E(4/3) - pc(5/3) \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1.533(5/3) + 1.445(4/3) \\ -1.533(4/3) - 1.445(5/3) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4.482 \\ -4.452 \\ 0 \\ 0 \end{pmatrix}.$$

We see that the total energy of the electron in the S' frame is $E' = 4.482$ MeV and its momentum is $p' = -4.452$ MeV/c. The negative sign in p' simply indicates that the electron moves to the left in the S' frame.

The energy E' and momentum p' can also be found by calculating the speed of the electron u' in S' using the relativistic velocity addition rules and then computing $E' = \gamma_{u'}mc^2$ and $p' = \gamma_{u'}mu'$.

(e) What is the speed of the electron u' relative to the speed of light in frame S'?

As above, we use the expressions for the relativistic 3-momentum ($p' = \gamma_{u'} m u'$) and total energy ($E' = \gamma_{u'} mc^2$):

$$\frac{p'c}{E'} = \frac{\gamma_{u'} m u' c}{\gamma_{u'} m c^2} = \frac{u'}{c} = \frac{4.452}{4.482} = 0.993.$$

So we see that in S', the electron's total energy, momentum, and velocity are all greater than in the laboratory frame. This makes sense since the electron and S' frame are moving in opposite directions.