PROBLEMS 10

(1) See Slides

(2)

1. Write the total energy equal to the sum of the potential energy and the kinetic energy.

$$E = K + U = \frac{1}{2}mv^2 - \frac{GM_E m}{r}$$

2. Apply Newton's second law to the satellite and solve for the square of the speed.

$$F = ma$$
$$\frac{GM_E m}{r^2} = m\frac{v^2}{r}$$
$$\text{so}\quad v^2 = \frac{GM_E}{r}$$

3. Substitute into the step-1 result and simplify.

$$E = \frac{1}{2}m\frac{GM_E}{r} - \frac{GM_E m}{r} = -\frac{GM_E m}{2r}$$

4. Compare the step-3 result with U in step 1.

$$E = -\frac{GM_E m}{2r} = \frac{1}{2}\left(-\frac{GM_E m}{r}\right) = \boxed{\frac{1}{2}U}$$

(3)

The speed of an object in an orbit of radius r around a planet is given by $v = \sqrt{GM_{\text{planet}}/r}$, and is also given by $v = 2\pi r/T$, where T is the period of the object in orbit. Equate the two expressions for the speed and solve for T.

$$\sqrt{G\frac{M_{\text{Planet}}}{r}} = \frac{2\pi r}{T} \quad\rightarrow\quad T = 2\pi\sqrt{\frac{r^3}{GM_{\text{Planet}}}}$$

For this problem, the inner orbit is at $r_{\text{inner}} = 7.3\times10^7\,\text{m}$, and the outer orbit is at $r_{\text{inner}} = 1.7\times10^8\,\text{m}$. Use these values to calculate the periods.

$$T_{\text{inner}} = 2\pi\sqrt{\frac{(7.3\times10^7\,\text{m})^3}{(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(5.7\times10^{26}\,\text{kg})}} = \boxed{2.0\times10^4\,\text{s}}$$

$$T_{\text{outer}} = 2\pi\sqrt{\frac{(1.7\times10^8\,\text{m})^3}{(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(5.7\times10^{26}\,\text{kg})}} = \boxed{7.1\times10^4\,\text{s}}$$

Saturn's rotation period (day) is 10 hr 39 min which is about 3.8×10^4 sec. Thus the inner ring will appear to move across the sky "faster" than the Sun (about twice per Saturn day), while the outer ring will appear to move across the sky "slower" than the Sun (about once every two Saturn days).

4.

(a) The speed of an object in near-surface orbit around a planet is given by $v=\sqrt{GM/R}$, where M is the planet mass and R is the planet radius. The speed is also given by $v=2\pi R/T$, where T is the period of the object in orbit. Equate the two expressions for the speed.

$$\sqrt{G\frac{M}{R}}=\frac{2\pi R}{T} \rightarrow G\frac{M}{R}=\frac{4\pi^2 R^2}{T^2} \rightarrow \frac{M}{R^3}=\frac{4\pi^2}{GT^2}$$

The density of a uniform spherical planet is given by $\rho=\frac{M}{\text{Volume}}=\frac{M}{\frac{4}{3}\pi R^3}$. Thus

$$\rho=\frac{3M}{4\pi R^3}=\frac{3}{4\pi}\frac{4\pi^2}{GT^2}=\frac{3\pi}{GT^2}$$

(b) For Earth,

$$\rho=\frac{3\pi}{GT^2}=\frac{3\pi}{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)[(85\ \text{min})(60\ \text{s/min})]^2}=\boxed{5.4\times10^3\ \text{kg/m}^3}$$

5.

Use Kepler's 3rd law to relate the orbits of Earth and Halley's comet around the Sun.

$$(r_{\text{Halley}}/r_{\text{Earth}})^3=(T_{\text{Halley}}/T_{\text{Earth}})^2 \rightarrow$$

$$r_{\text{Halley}}=r_{\text{Earth}}(T_{\text{Halley}}/T_{\text{Earth}})^{2/3}=(150\times10^6\text{km})(76\ \text{y}/1\ \text{y})^{2/3}=\boxed{2690\times10^6\text{km}}$$

This value is half the sum of the nearest and farthest distances of Halley's comet from the Sun. Since the nearest distance is very close to the Sun, we will approximate that nearest distance as 0. Then the farthest distance is twice the value above, or $5380\times10^6\text{km}$. This distance approaches the mean orbit distance of Pluto, which is $5900\times10^6\text{km}$. $\boxed{\text{It is still in the Solar System, nearest to Pluto's orbit.}}$

6.

(a) The relationship between satellite period T, mean satellite distance r, and planet mass M can be derived from the two expressions for satellite speed: $v=\sqrt{GM/r}$ and $v=2\pi r/T$. Equate the two expressions and solve for M.

$$\sqrt{GM/r}=2\pi r/T \rightarrow M=\frac{4\pi^2 r^3}{GT^2}$$

Substitute the values for Io to get the mass of Jupiter.

$$M_{\substack{\text{Jupiter-}\\ \text{Io}}}=\frac{4\pi^2(4.22\times10^8\text{m})^3}{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)\left(1.77\text{d}\times\frac{24\ \text{h}}{1\ \text{d}}\times\frac{3600\ \text{s}}{1\ \text{h}}\right)^2}=\boxed{1.90\times10^{27}\text{kg}}.$$

(b) For the other moons:

$$M_{\substack{\text{Jupiter-}\\ \text{Europa}}}=\frac{4\pi^2(6.71\times10^8\text{m})^3}{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(3.55\times24\times3600\ \text{s})^2}=\boxed{1.90\times10^{27}\text{kg}}$$

$$M_{\substack{\text{Jupiter-}\\ \text{Ganymede}}}=\frac{4\pi^2(1.07\times10^9\text{m})^3}{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(7.16\times24\times3600\ \text{s})^2}=\boxed{1.89\times10^{27}\text{kg}}$$

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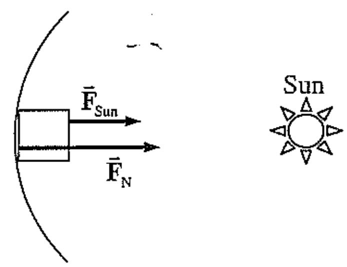
If the ring is to produce an apparent gravity equivalent to that of Earth, then the normal force of the ring on objects must be given by $F_N = mg$. The Sun will also exert a force on objects on the ring. See the free-body diagram.



Write Newton's 2nd law for the object, with the fact that the acceleration is centripetal.

$$\sum F = F_R = F_{Sun} + F_N = mv^2/r$$

Substitute in the relationships that $v = 2\pi r/T$, $F_N = mg$, and $F_{Sun} = G\dfrac{M_{Sun}m}{r^2}$, and solve for the period of the rotation.

$$F_{Sun} + F_N = mv^2/r \rightarrow G\frac{M_{Sun}m}{r^2} + mg = \frac{4\pi^2 mr}{T^2} \rightarrow G\frac{M_{Sun}}{r^2} + g = \frac{4\pi^2 r}{T^2}$$

$$T = \sqrt{\frac{4\pi^2 r}{G\frac{M_{Sun}}{r^2} + g}} = \sqrt{\frac{4\pi^2\left(1.50\times10^{11}\,\mathrm{m}\right)}{\left(6.67\times10^{-11}\,\mathrm{N{\bullet}m^2/kg^2}\right)\frac{\left(1.99\times10^{30}\,\mathrm{kg}\right)}{\left(1.50\times10^{11}\,\mathrm{m}\right)^2} + 9.8\,\mathrm{m/s^2}}} = 7.8\times10^5\,\mathrm{s} = \boxed{9.0\ \mathrm{d}}$$

The force of the Sun is only about 1/1600 the size of the normal force. The force of theSun could have been ignored in the calculation with noeffect in the result as given above.

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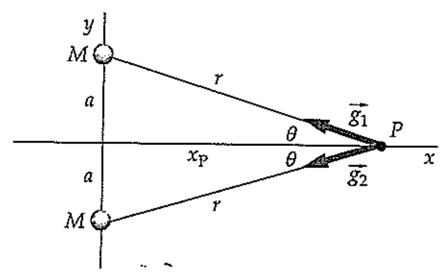

FIGURE 11-13 The particles are each located at a source point, and point P is a field point.

1. Calculate the magnitude of $\vec{g}_1$ and $\vec{g}_2$:

$$g_1 = g_2 = \frac{GM}{r^2}$$

2. The y component of the resultant field, the sum of g_{1y} and g_{2y}, is zero. The x component is the sum of g_{1x} and g_{2x}:

$$g_y = g_{1y} + g_{2y} = g_1 \sin\theta - g_2 \sin\theta = 0$$
$$g_x = g_{1x} + g_{2x} = g_1 \cos\theta + g_2 \cos\theta = 2g_1 \cos\theta = \frac{2GM}{r^2}\cos\theta$$

3. Express $\cos\theta$ in terms of x_P and r from the figure:

$$\cos\theta = \frac{x_P}{r}$$

4. Combining the last two results yields $\vec{g}$. To express $\vec{g}$ as a function of x_P, substitute $(x_P^2 + a^2)^{1/2}$ for r:

$$\vec{g} = g_x\hat{i} = -\frac{2GM}{r^2}\frac{x_P}{r}\hat{i} = -\frac{2GMx_P}{r^3}\hat{i} = -\frac{2GMx_P}{(x_P^2 + a^2)^{3/2}}\hat{i}$$

5. x_P is an arbitrary point on the x axis. For simplicity, we replace it with x:

$$\vec{g} = \boxed{-\frac{2GMx}{(x^2 + a^2)^{3/2}}\hat{i}}$$

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(*a*) 1. The mass M' (the mass of the spherical shell with outer radius $\frac{3}{4}R$ and inner radius $\frac{1}{2}R$) is the density ρ times the volume V':

$$M' = \rho V'$$

2. The density is the total mass M divided by the total volume V:

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3 - \frac{4}{3}\pi(\frac{1}{2}R)^3} = \frac{M}{\frac{7}{6}\pi R^3} = \frac{6M}{7\pi R^3}$$

3. Find the volume V' of the thick shell with outer radius $\frac{3}{4}R$ and inner radius $\frac{1}{2}R$:

$$V' = \frac{4}{3}\pi\left(\frac{3R}{4}\right)^3 - \frac{4}{3}\pi\left(\frac{R}{2}\right)^3 = \frac{19}{48}\pi R^3$$

4. Find the mass M':

$$M' = \rho V' = \frac{6M}{7\pi R^3}\frac{19}{48}\pi R^3 = \boxed{\frac{19}{56}M}$$

(*b*) The gravitational field at $r = \frac{3}{4}R$ is due only to the mass M':

$$\vec{g} = -\frac{GM'}{r^2}\hat{r} = -\frac{G\frac{19}{56}M}{(\frac{3}{4}R)^2}\hat{r} = \boxed{-\frac{38}{63}\frac{GM}{R^2}\hat{r}}$$

10

(*a*) 1. Integrate $dM = \rho\, dV$ to relate C to the mass M, where $dV = 4\pi r^2\, dr$. ($4\pi r^2$ is the area of a sphere of radius r so $4\pi r^2\, dr$ is the volume of a spherical shell of radius r and thickness dr):

$$M = \int dM = \int \rho\, dV$$
$$= \int_0^R Cr(4\pi r^2\, dr) = C\pi R^4$$

2. Solve for C in terms of the given quantities M and R.

$$C = \boxed{\frac{M}{\pi R^4}}$$

(*b*) Write an expression for the field outside the sphere in terms of the mass M, the distance r from the center, and the unit vector $\hat{r}$. The unit vector $\hat{r}$ is in the direction of increasing r:

$$\vec{g} = \boxed{-\frac{GM}{r^2}\hat{r}} \quad (r > R)$$

(*c*) 1. Compute the mass M' that is within the radius $\frac{1}{2}R$ by integrating $dm = \rho\, dV$ from $r = 0$ to $\frac{1}{2}R$ and use the value of C found in Part (*a*), step 2.

$$M' = \int \rho\, dV = \int_0^{R/2} Cr(4\pi r^2\, dr) = C\pi R^4/16$$
$$M' = \frac{M}{16}$$

2. Write an expression for the field at $r = \frac{1}{2}R$ in terms of M and R.

$$\vec{g} = -\frac{GM'}{r^2}\hat{r} = \boxed{-\frac{GM}{4R^2}\hat{r} \quad \text{at} \quad r = \tfrac{1}{2}R}$$

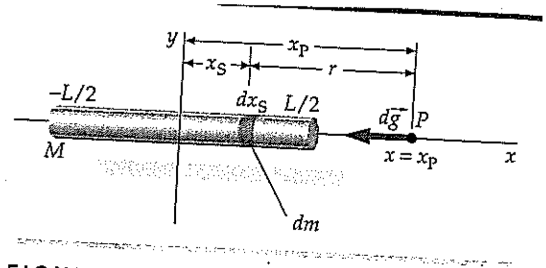

FIGURE [redacted] All points on x axis in the range $-L/2 < x < L/2$ are source points, and point P is a field point.

1. Find the x component of the field at P due to the mass element dm: $\quad dg_x = -\dfrac{G\,dm}{r^2}$

2. Because the rod is uniform, the mass per unit length λ is constant and equal to the total mass divided by the total length. The mass dm of an element of length dx_S is equal to the mass per unit length times the length dx_S: $\quad dm = \lambda\,dx$ where $\lambda = \dfrac{M}{L}$

3. Write the distance r between dm and point P in terms of x_S and x_P: $\quad r = x_P - x_S$

4. Substitute these results to express dg in terms of x:
$$dg_x = -\frac{G\,dm}{r^2} = -\frac{G\lambda\,dx_S}{(x_P - x_S)^2}$$

5. Integrate to find the x component of the resultant field:
$$g_x = \int dg_x = -G\lambda\int_{-L/2}^{L/2}\frac{dx_S}{(x_P - x_S)^2} = -\frac{GM}{x_P^2 - (L/2)^2}$$

6. Express the resultant field as a vector:
$$\vec{g} = g_x\hat{i} = -\frac{GM}{x_P^2 - (L/2)^2}\hat{i}$$

7. Here x_P is an arbitrary point on the x axis in the region $x > L/2$. For simplicity, we replace it with x:
$$\vec{g} = \boxed{-\frac{GM}{x^2 - (L/2)^2}\hat{i} \qquad x > L/2}$$

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The speed of rotation of the Sun about the galactic center, under the assumptions made, is given by $v = \sqrt{G\dfrac{M_{galaxy}}{r_{Sun\ orbit}}}$ and so $M_{galaxy} = \dfrac{r_{Sun\ orbit}v^2}{G}$. Substitute in the relationship that $v = 2\pi r_{Sun\ orbit}/T$.

$$M_{galaxy} = \frac{4\pi^2(r_{Sun\ orbit})^3}{GT^2} = \frac{4\pi^2\left[(30{,}000)(9.5\times10^{15}\,\mathrm{m})\right]^3}{(6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2})\left[(200\times10^6\,\mathrm{y})\left(\dfrac{3.15\times10^7\,\mathrm{s}}{1\,\mathrm{y}}\right)\right]^2}$$

$$= 3.452\times10^{41}\,\mathrm{kg} \approx \boxed{3\times10^{41}\,\mathrm{kg}}$$

The number of solar masses is found by dividing the result by the solar mass.

$$\#\ \text{stars} = \frac{M_{galaxy}}{M_{Sun}} = \frac{3.452\times10^{41}\,\mathrm{kg}}{2.0\times10^{30}\,\mathrm{kg}} = 1.726\times10^{11} \approx \boxed{2\times10^{11}\,\text{stars}}$$