- **Centripetal acceleration**

  **- Acceleration perpendicular to the velocity**

  For any point of a smooth curve one can define curvature center and curvature radius.

  In general there are both centripetal acceleration and acceleration along the velocity.

  \[ \mathbf{a} \perp \mathbf{v} \text{ – centripetal acceleration, directed toward the curvature center} \]

Let \(|\mathbf{v}|=\text{const}\). Then \(\mathbf{v}\) can only change direction.

**Magnitude of the centripetal acceleration**

From similarity of triangles:

\[ \frac{\Delta \mathbf{v}}{\mathbf{v}} = \frac{\Delta r}{R} \]

Therefore:

\[ a_c = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v}}{R} \frac{\Delta r}{\Delta t} = \frac{\mathbf{v}}{R} \mathbf{v} = \frac{\mathbf{v}^2}{R} \]
Angles $\theta$ and $\Delta \theta$ can be measured in:
- degrees
- revolutions (360°) (used in engineering)
- radians (used in physics)

In radians, $\theta = L/R$ (no unit!)

Full circle: $\theta = 2\pi R/R = 2\pi = 360°$

Thus

$$1 \text{ radian} = \frac{360°}{2\pi} = 57.3°$$

Useful formulas (for $\theta$ in radians):

- $\sin \theta \cong \tan \theta \cong \theta$
- $\cos \theta \cong 1 - \theta^2/2$
- for $\theta \ll 1$
Angular velocity is the rate of change of the angle with time: \[
\omega = \frac{\Delta \theta}{\Delta t}
\]

Displacement due to rotation

If a vector \( \mathbf{R} \) is rotated by a small angle \( \Delta \theta \), the change of the vector (the displacement of its end point) \( \Delta \mathbf{r} \) is proportional to \( \Delta \theta \), so that \( \Delta \mathbf{r} = R \Delta \theta \). This can be derived by approximating the small arc by a straight line and considering the two triangles with one angle equal to 90°, as shown:

\[
\Delta \mathbf{r} = 2R \sin \frac{\Delta \theta}{2} \approx R \Delta \theta \quad \text{for} \ \theta \ll 1
\]

Angular velocity

Relation between the angular and linear velocities

can be derived using the displacement-rotation relation above:

\[
v = \frac{\Delta r}{\Delta t} = \frac{R \Delta \theta}{\Delta t} = R \frac{\Delta \theta}{\Delta t} = R \omega,
\quad \text{or} \quad \omega = \frac{v}{R}
\]

Thus \( a_c = \frac{v^2}{R} = \frac{(R \omega)^2}{R} = \omega^2 R \) - another formula for the centripetal acceleration.
Angular velocity, frequency, period

The angular velocity $\omega$ is defined as

$$\omega = \frac{\Delta \theta}{\Delta t},$$

where $\theta$ is the rotation angle in radians. The frequency of rotations $f$ is defined as the number of rotations per second,

$$f = \frac{\text{number of rotations}}{\Delta t} \quad \text{(special unit: Hertz (Hz))}.$$ 

As one rotation corresponds to $2\pi$ radians, the number of rotations in the angle $\Delta \theta$ is given by $\Delta \theta/(2\pi)$. Thus

$$f = \frac{\Delta \theta / (2\pi)}{\Delta t} = \frac{1}{2\pi} \frac{\Delta \theta}{\Delta t} = \frac{\omega}{2\pi},$$

and $\omega = 2\pi f$. The period $T$ of rotations is defined as the time needed for one rotation, that is,

$$T = \frac{\Delta t}{\text{number of rotations}} = \frac{1}{f} = \frac{2\pi}{\omega}.$$
- Forces that create centripetal acceleration such as tension of a string, friction of tires against the road, etc., play the role of centripetal forces.

\[ F_c = m a_c \]

\[ F_c = m \frac{v^2}{R} \]

A car on a curved road: friction force \( F_{fr} \) plays the role of the centripetal force

A stone on a string: tension force \( T \) plays the role of the centripetal force

\[ F_{fr} = m \frac{v^2}{R} \leq \mu_s F_N = \mu_s mg \]

Driving is possible if the traction condition \( v \leq \sqrt{\mu_s Rg} \) is satisfied, otherwise skidding
**Ferris wheel**

At lower point:

\[ F_N - mg = ma_c \]

\[ \Rightarrow F_N = m(g + a_c) \]

Larger pressure on seat (apparent weight)

At upper point:

\[ F_N - mg = -ma_c \]

\[ \Rightarrow F_N = m(g - a_c) \]

Smaller pressure on seat (apparent weight)
Conic pendulum (rotating with the angular velocity $\omega$ around the vertical axis)

$L$ – length of the pendulum, $T$ – tension of the string

$T_x = T \sin \theta$ is the centripetal force

Newton’s second law:

"$y$": $T \cos \theta - mg = 0 \rightarrow \theta = \arccos \frac{mg}{T}$

"$x$": $T \sin \theta = F_c = m\omega^2R = m\omega^2 L \sin \theta \rightarrow T = m\omega^2 L$ (if $\sin \theta \neq 0$)

Thus $\theta = \arccos \frac{mg}{m\omega^2 L} = \arccos \frac{g}{\omega^2 L} = \arccos \frac{\omega_c^2}{\omega^2}$,
under the condition $\omega \geq \omega_c = \sqrt{g/L}$.

For $\omega \leq \omega_c$ the solution is $\theta = 0$, $T = mg$ (check!)
Car on a banked road

Both friction force and normal force contribute to $F_c$:

\[
\begin{align*}
\text{“Z“} & : -mg \cos\theta + F_N = m\frac{v^2}{R} \\
\text{“X“} & : mg \sin\theta + F_{fr} = m\frac{v^2}{R} \cos\theta
\end{align*}
\]

Optimal angle: $F_{fr} = 0$  \quad \tan\theta = \frac{v^2}{Rg}$

Traction condition: $F_{fr} \leq \mu_s F_N$

\[
\begin{align*}
-v^2 \cos\theta - \mu_s \sin\theta & \leq \mu_s \left(g \cos\theta + \frac{v^2}{R} \sin\theta\right) \\
v^2 (1 - \mu_s \tan\theta) & \leq gR (\tan\theta + \mu_s)
\end{align*}
\]

\[
v^2 \leq gR \frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta} \quad \text{for} \quad \mu_s \tan\theta < 1
\]

For $\mu_s \tan\theta > 1$ the traction condition is satisfied for any speed!
Evolution of views on planetary motion

- Ancient Greeks
  Naïve geocentric system
  Also heliocentric system!

- Claudius Ptolemaeus (87-150, Egypt)
  Elaborate geocentric system with math methods and epicycles

- Nicolaus Copernicus (1473-1543, Poland)
  Revived the heliocentric system
  Championed heliocentric system by Copernicus, built telescopes

- Galileo Galilei (1564-1642, Italy)
  Collected lots of high-accuracy data
  On planet motion (without telescopes)

- Tycho Brahe (1546-1601, Danmark)
  Obtained 3 laws of planetary motion from analysis of Tycho's data

- Johannes Kepler (1571-1630, Germany)
  Obtained the law of gravitation from Kepler's laws

- Isaac Newton (1642-1727, England)
Planetary motion: Kepler’s laws

1. The orbits of planets are ellipses with the sun in one of the focuses.

2. The radius vector sweeps out equal areas in equal time.

3. \( \frac{T^2}{R^3} = \text{const} \) for all planets of our solar system.
   - \( T \) – period of the motion
   - \( R \) – average distance from the sun

(particular case: circle)
Newton’s law of universal gravitation

Planet motion - projectile motion!

Falling down

Gravitational force should decrease with distance

Newton showed mathematically that Kepler’s laws follow from the second Newton’s law with

\[ F = \frac{Gm_1m_2}{r^2} \]

Experiments by Cavendish (1731-1810)

\[ G = 0.667 \times 10^{-10} \text{ N} \cdot \text{m}^2 / \text{kg}^2 \]

Mass of the Earth:

\[ m_E = \frac{gR_E^2}{G} \approx 0.6 \times 10^{25} \text{ kg} \]

with \( R_E = 6400 \text{ km} \)
Special case of Kepler’s third law for circular orbits

\[ m \approx \text{mass of the Earth, } M \approx \text{mass of the Sun} \]

\[ G \frac{mM}{R^2} = m \frac{v^2}{R} \quad (m \ll M) \]

\[ v^2 = G \frac{M}{R} \quad \text{Also, } v = \frac{2\pi R}{T} \]

Eliminating \( v \) from these two equations, one obtains

\[ G \frac{M}{R} = \left( \frac{2\pi R}{T} \right)^2 \]

\[ \frac{T^2}{R^3} = \frac{4\pi^2}{GM} \]

\( (T = 1 \text{ year}, R = 1.5 \times 10^{11} \text{ m} \Rightarrow M = 2.0 \times 10^{30} \text{ kg}) \)

Then for two objects rotating around the same center

\[ \frac{T_1^2}{R_1^3} = \frac{T_2^2}{R_2^3} \]

- the third Kepler’s law

It is more convenient to derive the 3-rd Kepler’s law using the angular velocity

\[ G \frac{mM}{R^2} = m \omega^2 R \]

\[ \omega^2 R^3 = GM \]

- this is already the third Kepler’s law!

Using \( \omega = \frac{2\pi}{T} \), one rewrites it as

\[ \frac{R^3}{T^2} = \frac{GM}{4\pi^2} \]

Geosynchronous satellites

A satellite can have the period of its orbiting equal to that of the rotation of the Earth over its axis. If the satellite is orbiting in the equatorial plane, that it will have the same position on the sky. For this, the orbit radius should have a particular value

\[ R = \left( \frac{T^2}{4\pi^2 GM} \right)^{1/3} \]

With \( T = 1 \text{ day} = 24 \times 3600 = 86400 \text{ s} \) and \( M = 0.6 \times 10^{25} \text{ kg} \) one obtains

\[ R = \left( \frac{86400^2}{4\pi^2} \times 0.667 \times 10^{-10} \times 0.6 \times 10^{25} \right)^{1/3} = 4.23 \times 10^7 \text{ m} = 42300 \text{ km}. \]