6 – Work and Energy

**Work** - The concept following from the analysis of simple mechanical devices

- **Lever**
  - Simple machines allow to gain in force: achieve a greater *output* force with a smaller *input* force
  - Gaining in force one loses in distance
    \[ F_1 d_1 = F_2 d_2 \]

- **Pulley**
  - Concept of *work*:
    \[ \text{Work} = \text{force} \times \text{distance} \] is conserved: \((\text{work input}) = (\text{work output})\)
**Definition of Work for a constant force**

![Diagram of force and displacement vectors]

Definition: \( W = F_x d = Fd \cos \theta = \mathbf{F} \cdot \mathbf{d} \)  
(dot-product of two vectors)

Force components perpendicular to displacement do not produce work as \( \cos \theta = 0 \)!  
Work is negative, if \( \cos \theta < 0 \)

Unit of work: J(oule)  
\( J = \text{N(ewton)} \cdot \text{m} = \text{kg} \cdot \text{m}^2/\text{s}^2 \)

**Example:**

![Diagram of example scenario]

**Problem:**

- \( m = 50 \text{ kg}, \ F = 100 \text{ N}, \ F_{\text{fr}} = 50 \text{ N} \)
- \( \theta = 37^\circ, \ d = 40 \text{ m} \)

Work done by each force - ?

**Solution**

\( W_G = mgd \cos 90^\circ = 0 \), \( W_N = F_N d \cos 90^\circ = 0 \)

\( W_{\text{Pull}} = Fd \cos \theta = 100 \text{ N} \times 40 \text{ m} \times \cos 37^\circ = 3200 \text{ J} \)

\( W_{\text{fr}} = F_{\text{fr}} d \cos 180^\circ = 50 \text{ N} \times 40 \text{ m} \times (-1) = -2000 \text{ J} \)
\textit{Infinite\textit{esimal work and total work}}

If the force is not constant but changes from point to point, one has to consider the \textit{infinitesimal} work corresponding to an \textit{infinitesimal} displacement $\Delta \mathbf{r} \to 0$:

$$\Delta W = \mathbf{F} \cdot \Delta \mathbf{r}$$

The total work is the sum of all infinitesimal works along the trajectory:

$$W = \sum_i \Delta W_i = \sum_i \mathbf{F}_i \cdot \Delta \mathbf{r}_i$$

\textbf{Power} \hspace{1cm} \text{- Rate of doing work}

Instantaneous power: $P = \frac{\Delta W}{\Delta t}$

Unit of power: $W(\text{att})$

It can be expressed as $P = \frac{\mathbf{F} \cdot \Delta \mathbf{r}}{\Delta t} = \mathbf{F} \cdot \mathbf{v}$

$$W = J(\text{oule}) / s = \text{kg m}^2 / \text{s}^3$$

\textbf{Car:}

For $P = \text{const}$ one has $F = \frac{P}{v}$, so that the maximal acceleration $a = \frac{F}{m} = \frac{P}{mv}$ decreases with the speed
(Mechanical) Energy - Work stored in a body or ability of a body to do work

Mechanical energy = Kinetic energy + Potential energy

\[ E = E_{\text{kin}} + E_{\text{pot}} \]

\[ E_{\text{kin}} = \frac{mv^2}{2}, \quad E_{\text{pot}} - \text{different forms} \]

Work done \quad Increase of energy

Illustration for the linear motion with constant acceleration \((x_0 = v_0 = 0)\)

Work (function of the process):

\[ W = Fx = ma \frac{1}{2} at^2 = m(at)^2 = \frac{mv^2}{2} \quad \text{Kinetic energy} \]

In general:

\[ W_{12} = \Delta E = E_2 - E_1 \]

Work of external forces done on the way from position 1 to position 2 equals the change of energy of the system

Or \( W = E_f - E_i \), where \( E_f \) and \( E_i \) are the final and initial total energies and \( W \) is the work of the external forces on the system.
Potential Energy

Work of the external force needed to bring a system from the reference state into another state quasistatically ($v \to 0$)

Potential energy is defined up to an arbitrary constant that can be understood as the potential energy of the reference state

Gravitational energy

\[ E_{pot} = W = Fh = mgh \]

Example: free fall from the height $h$

and the energy conservation (see next slides).

Prove that the total energy is conserved, $E_f = E_i$.

Proof:

Initial state: $E_{pot} = mgh$ and $E_{kin} = 0$, $E_i = mgh$

Final state: $z = 0$, $E_{pot} = mgz = 0$, $E_{kin} = \frac{mv^2}{2}$

\[ v = -gt, \quad z = h - \frac{1}{2}gt^2, \quad z = 0 \Rightarrow t^2 = \frac{2h}{g} \]

\[ E_f = E_{kin} = \frac{mv^2}{2} = \frac{mg^2t^2}{2} = \frac{mg^22h/g}{2} = mgh = E_i. \]

OK
Elastic energy (the energy of a deformed spring)

Hooke's law for the spring: \( F_{Hooke} = -kx \)

- \( k \) – stiffness of the spring
- \( x \) – elongation/kompression, \( x \equiv X - X_0 \)
- \( X_0 \) - the length of the free spring
- \( X \) – the length of the deformed spring

\[
F + F_{Hooke} = ma = 0 \rightarrow F = kx \text{ (external force)}
\]

\[
E_{pot} = W = \sum_i F(x_i)\Delta x_i \text{ - area under the curve } F(x)
\]

\[
E_{pot} = \frac{1}{2} kx^2 \]

\( F \) vs. \( x \)
Conservation of Energy

In the absence of dissipation (friction) the total energy of an isolated system is conserved:

\[ E = E_{\text{tot}} = E_{\text{pot}} + E_{\text{kin}} = \text{const} \]

Energies of the two different kinds can be transformed into each other:
- potential energy can be released into kinetic energy
- kinetic energy can be absorbed into potential energy

**Problem**
At 1: \( m=0.5 \text{ kg}, \ h=12 \text{ cm}, \ v=0 \)
Speed at 2 ?

**Solution**
Energy conservation in general:
\[ E_1 = E_{\text{kin},1} + E_{\text{pot},1} = E_2 = E_{\text{kin},2} + E_{\text{pot},2} \]
But here \( E_{\text{kin},1} = E_{\text{pot},2} = 0 \),
thus, the energy conservation has the form
\[ E_{\text{pot},1} = mgh = E_{\text{kin},2} = \frac{mv^2}{2} \]
\[ v_2 = \sqrt{2gh} = \sqrt{2 \times 9.8 \text{ m/s}^2 \times 0.12 \text{ m}} = 1.53 \text{ m/s} \]
Problem

A dart of a mass 0.100 kg is pressed against the spring of a toy dart gun. The spring with spring stiffness \( k = 250 \text{ N/m} \) is compressed 6.0 cm and released. If the dart detaches from the spring when the spring is reaching its natural length \((x=0)\) what speed does the dart acquire?

\[
\text{Known: } m = 0.1 \text{ kg}, \quad k = 250 \text{ N/m}, \quad x_1 = 6 \text{ cm} = 0.06 \text{ m}
\]

\[
\text{To find: } v_2 - ?
\]

Solution: The total energy of the system spring + dart is conserved

State 1: Deformed spring, potential energy

State 2: Flying dart, kinetic energy

\[
E_1 = E_2 \quad \Rightarrow \quad \frac{1}{2} kx_1^2 = \frac{mv_2^2}{2} \quad \Rightarrow \quad v_2 = \sqrt{\frac{kx_1^2}{m}} = x_1 \sqrt{\frac{k}{m}}
\]

\[
\sqrt{x^2} = (x^2)^{1/2} = x^{2\times1/2} = x^1 = x
\]

More accurately:

\[
\sqrt{x^2} = |x|
\]

Plugging numbers:

General analytical result

\[
v_2 = 0.06 \text{ m} \sqrt{\frac{250 \text{ N/m}}{0.1 \text{ kg}}} = 0.06 \sqrt{2500} = 0.06 \times 50 = 3 \text{ m/s}
\]

Check units separately:

\[
m \sqrt{\frac{\text{N/m}}{\text{kg}}} = m \sqrt{\frac{\text{kg m/s}^2}{\text{m}}} = m \sqrt{1/s^2} = \text{m/s}, \quad \text{OK}
\]