Analytical Dynamics - Graduate Center CUNY - Fall 2008
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Rotational motion of rigid bodies

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1 Rotational kinematics

1.1 Large and small rotations

Rotations of rigid bodies are described with respect to their center of mass (CM) located at O or with respect to any other point O', in particular, a support point or a point on the so-called "instantaneous axis of rotation". In the latter case, the point O' is not connected to the body and moves with respect to body's particles.

The orientation of a rigid body is defined by the directions of its own embedded Descarte axes $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$ with respect to the laboratory Descarte axes $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_x$. Any rotation of a body out of the reference orientation $\mathbf{e}^{(\alpha)} = \mathbf{e}_\alpha$, $\alpha = x, y, z = 1, 2, 3$, to an arbitrary orientation can be described by a rotation by a finite angle χ around some axis $\boldsymbol{\nu}$. This is the Euler theorem obtained by geometrical arguments [L. Euler, Novi Comment. Petrop. XX, 189 (1776)]. As the direction of $\boldsymbol{\nu}$ is specified by two angles ϑ and φ of the spherical coordinate system, this finite rotation is specified by three angles ϑ, φ, and χ.

In the finite rotation described above, any embedded vector $\mathbf{r}$, including $\mathbf{e}^{(\alpha)}$, is rotated as

$$\begin{aligned} \mathbf{r} \quad &\rightarrow \quad \mathbf{r}' = (\boldsymbol{\nu}\cdot\mathbf{r})\,\boldsymbol{\nu} - [\boldsymbol{\nu}\times[\boldsymbol{\nu}\times\mathbf{r}]]\cos\chi + [\boldsymbol{\nu}\times\mathbf{r}]\sin\chi \\ &= \quad \mathbf{r}\cos\chi + \boldsymbol{\nu}\,(\boldsymbol{\nu}\cdot\mathbf{r})\,(1-\cos\chi) + [\boldsymbol{\nu}\times\mathbf{r}]\sin\chi. \end{aligned} \tag{1}$$

The first line of this equation tells the story how it was obtained. Vector $(\boldsymbol{\nu}\cdot\mathbf{r})\,\boldsymbol{\nu}$ is the projection of $\mathbf{r}$ on $\boldsymbol{\nu}$ (the longitudinal componet of $\mathbf{r}$) that remains invariant under rotation. Vectors $[\boldsymbol{\nu}\times[\boldsymbol{\nu}\times\mathbf{r}]]$ and $[\boldsymbol{\nu}\times\mathbf{r}]$ are perpendicular to $\boldsymbol{\nu}$ and to each other, thus they form a basis in the plane perpendicular to $\boldsymbol{\nu}$. The first of them is just the transverse component of $\mathbf{r}$. The transverse component of $\mathbf{r}'$ that is rotated by χ is projected on these two basis vectors. To obtain the second line, we used the identity

$$[\mathbf{a}\times[\mathbf{b}\times\mathbf{c}]] = \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\,(\mathbf{a}\cdot\mathbf{b})\,. \tag{2}$$

For small rotation angles $\delta\chi$ at linear order in $\delta\chi$ this transformation simplifies to

$$\mathbf{r}\rightarrow\mathbf{r}' \cong \mathbf{r} + [\boldsymbol{\nu}\times\mathbf{r}]\,\delta\chi = \mathbf{r} + [\delta\boldsymbol{\chi}\times\mathbf{r}]\,, \tag{3}$$

where the vector rotation angle $\delta\boldsymbol{\chi}\equiv\boldsymbol{\nu}\delta\chi$ was introduced. Note that the latter is possible only for small rotation angles. First-order Eq. (3) is sufficient to introduce the angular velocity below. Note that the axis $\boldsymbol{\nu}$ changes during the rotation as well but the corresponding terms are bilinear in $\delta\chi$ and $\delta\boldsymbol{\nu}$ and can be neglected.

The result of two and more successive rotations by large angles using Eq. (1) depend on the order in which rotations are performed. Indeed, Eq. (1) is a linear transformation that can be represented by a matrix, and matrices in general do not commute. However, for small rotations at linear order in $\delta\chi$ the result does not depend on the order of rotations. As an example consider two successive rotations

$$\mathbf{r}^{(1)} \cong \mathbf{r} + \left[\delta\boldsymbol{\chi}^{(1)}\times\mathbf{r}\right], \qquad \mathbf{r}^{(2)} \cong \mathbf{r}^{(1)} + \left[\delta\boldsymbol{\chi}^{(2)}\times\mathbf{r}^{(1)}\right]. \tag{4}$$

Combining these two formulas and keeping only terms linear in rotation angles, one obtains

$$\begin{aligned} \mathbf{r}^{(2)} \quad &\cong \quad \mathbf{r} + \left[\delta\boldsymbol{\chi}^{(1)}\times\mathbf{r}\right] + \left[\delta\boldsymbol{\chi}^{(2)}\times\left(\mathbf{r} + \left[\delta\boldsymbol{\chi}^{(1)}\times\mathbf{r}\right]\right)\right] \\ &\cong \quad \mathbf{r} + \left[\left(\delta\boldsymbol{\chi}^{(1)} + \delta\boldsymbol{\chi}^{(2)}\right)\times\mathbf{r}\right] \end{aligned} \tag{5}$$

This result is very important because one can consider components of $\delta\boldsymbol{\chi}$ without taking care of their order and one can treat more complicated rotations (such as rolling of a cone) as a superposition of two or more small rotations.

1.2 Angular velocity

The change of any vector $\mathbf{r}$ embedded in a rigid body due to an infinitesimal rotation and the corresponding velocity follow from Eq. (3):

$$\delta\mathbf{r} = \mathbf{r}' - \mathbf{r} \cong [\delta\boldsymbol{\chi} \times \mathbf{r}] \tag{6}$$

and

$$\mathbf{v} = \frac{\delta\mathbf{r}}{\delta t} = [\boldsymbol{\omega} \times \mathbf{r}], \tag{7}$$

where the angular velocity $\boldsymbol{\omega}$ is defined by

$$\boldsymbol{\omega} = \frac{\delta\boldsymbol{\chi}}{\delta t} \tag{8}$$

The definition of $\boldsymbol{\omega}$ is written in this form (and not as $\boldsymbol{\omega} = d\boldsymbol{\chi}/dt$) since the function $\boldsymbol{\chi}(t)$ that could be differentiated over time does not exist, in general. General rotations are not described by a single rotation vector because as the body rotates, the direction of the rotation axis changes, too. This makes rotational dynamics complicated.

Eq. (7) is valid in the case of the origin of the coordinate system for $\mathbf{r}$ fixed in space, such as the case of rotation around a point of support (fulcrum). If the origin is moving with velocity $\mathbf{V}$, one has to add this motion to the rotational motion and write a more general formula

$$\mathbf{v} = \mathbf{V} + [\boldsymbol{\omega} \times \mathbf{r}]. \tag{9}$$

In this case the origin of the coordinate system is usually put into the center of mass.

One can find the acceleration of any point of a body due to its rotation. Differentiating Eq. (7) (in the sence discussed above), substituting it again in the result, and using Eq. (2) yields

$$\begin{aligned} \dot{\mathbf{v}} &= [\dot{\boldsymbol{\omega}} \times \mathbf{r}] + [\boldsymbol{\omega} \times \dot{\mathbf{r}}] = [\dot{\boldsymbol{\omega}} \times \mathbf{r}] + [\boldsymbol{\omega} \times [\boldsymbol{\omega} \times \mathbf{r}]] \\ &= [\dot{\boldsymbol{\omega}} \times \mathbf{r}] + \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \mathbf{r}) - \mathbf{r}\omega^2 = [\dot{\boldsymbol{\omega}} \times \mathbf{r}] - \left(\mathbf{r} - \frac{\boldsymbol{\omega}(\boldsymbol{\omega} \cdot \mathbf{r})}{\omega^2}\right)\omega^2 \\ &= [\dot{\boldsymbol{\omega}} \times \mathbf{r}] - \mathbf{r}_\perp \omega^2. \end{aligned} \tag{10}$$

Here $\mathbf{r}_\perp$ is the component of $\mathbf{r}$ in the direction perpendicular to the rotation axis $\boldsymbol{\omega}$ since the component of $\mathbf{r}$ parallel to $\boldsymbol{\omega}$ was subtracted. The second term in Eq. (10) is the centripetal acceleration, whereas the first term is due to the angular acceleration.

1.3 Rolling constraint, instantaneous axis of rotation

Now consider Eq. (9) valid in the coordinate system with the origin O and introduce another origin O' by $\mathbf{a} = \text{const}$ away from O. The position vectors $\mathbf{r}$ and $\mathbf{r}'$ in both coordinate systems are related by $\mathbf{r} = \mathbf{a} + \mathbf{r}'$. Thus the velocity of the same physical point of the body with respect to another coordinate system reads

$$\mathbf{v} = \mathbf{V} + [\boldsymbol{\omega} \times \mathbf{a}] + [\boldsymbol{\omega} \times \mathbf{r}']. \tag{11}$$

If, for instance a sphere of radius R is rolling on a surface without slipping, it is convenient to place O' at the contact point between the sphere and the surface. In this case $\mathbf{a} = -\mathbf{n}R$, where $\mathbf{n}$ is the unit vector normal to the surface and directed outside, towards the center of the sphere. The velocity of the point of the sphere ($\mathbf{r}' = 0$) that is in contact with the surface at any moment of time is zero. This constraint condition can be written as

$$\mathbf{V} = -[\boldsymbol{\omega} \times \mathbf{a}] = [\boldsymbol{\omega} \times \mathbf{n}]R, \tag{12}$$

and then from Eq. (11) one obtains

$$\mathbf{v} = [\boldsymbol{\omega} \times \mathbf{r}'] \,. \tag{13}$$

This is similar to Eq. (7) and describes rotation around a horizontal instantaneous axis that goes through the contact point. It is called instantaneous not only because it can change its direction with time (for sphere or disc but not for cylinder) but because different points of the surface of the sphere are in contact with the support surface at different moments of time. This can lead to some misunderstanding and confusion since it is clear that two types of motion, rolling on a surface and rotation around a point of support are not the same. Nevertheless, *infinitesimally small* displacements of the body (at first order in δt) and hence the velocities are the same in both cases. In the second order in δt the two types of motion differ, so that the accelerations of different points of the sphere cannot be correctly obtained considering rotation around a fixed contact point, i.e., by differentiating Eq. (13) over time.

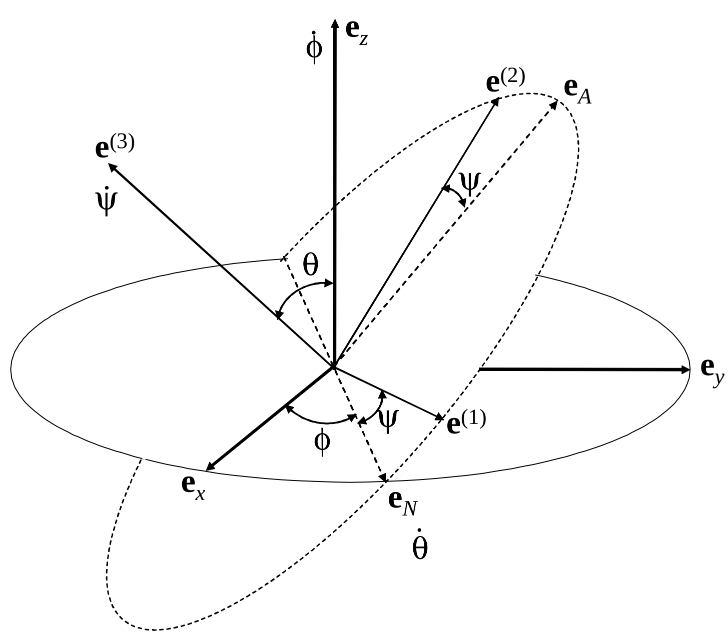
In the case of a disc rolling on a plane the constraint equation is different from Eq. (12) and has the form

$$\mathbf{V} = [\boldsymbol{\omega} \times \mathbf{e}_O] \, R, \tag{14}$$

where $\mathbf{e}_O$ is a unit vector pointing from O' to O that is not perpendicular to the plane and depends on the orientation of the disc. In the case of an ellipsoid rolling on a plane the constraint equation becomes more complicated.

The rolling constraints above for the sphere and disk are non-holonomic because they cannot be integrated eliminating generalized velocities and resulting in a constraint for a generalized coordinates. Only for a cylinder that is described by a single dynamical variable φ this can be done and thus the rolling constraint is holonomic.

1.4 Euler angles



In Sec. 1.1 we have seen that any orientation of the embedded coordinate system of a rigid body $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$ with respect to the laboratory frame $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ can be described by a sigle finite rotation specified by three angles. However, in mechanics of rigid bodies orientations are described as a result of three successive elementary rotations by Euler angles θ, ϕ, and ψ shown in the Figure. Rotation of the body gives rise to the three constituents of the angular velocity:

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{e}_N + \dot{\phi}\mathbf{e}_z + \dot{\psi}\mathbf{e}^{(3)}, \tag{15}$$

where $\mathbf{e}_N$ is the unit vector along the line of nodes ON. In addition to the auxiliary node vector $\mathbf{e}_N$ we will need the "antinode" vector $\mathbf{e}_A$ that lies in the plane spanned by $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ and is perpendicular to both $\mathbf{e}_N$ and $\mathbf{e}^{(3)}$. Using

$$\begin{aligned} \mathbf{e}_N &= \mathbf{e}^{(1)}\cos\psi - \mathbf{e}^{(2)}\sin\psi \\ \mathbf{e}_z &= \mathbf{e}_A\sin\theta + \mathbf{e}^{(3)}\cos\theta, \\ \mathbf{e}_A &= \mathbf{e}^{(1)}\sin\psi + \mathbf{e}^{(2)}\cos\psi \end{aligned} \tag{16}$$

one can project $\boldsymbol{\omega}$ onto the internal frame:

$$\begin{aligned} \omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 &= \dot{\phi}\cos\theta + \dot{\psi}. \end{aligned} \tag{17}$$

One speaks about the motion $\dot{\theta}$ as nutation, $\dot{\phi}$ as precession (for prolate bodies) or wobble (for oblate bodies), and $\dot{\psi}$ as a spin. One also can project $\boldsymbol{\omega}$ of Eq. (15) onto the laboratory axes:

$$\begin{aligned} \omega_x &= \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi \\ \omega_y &= -\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi \\ \omega_z &= \dot{\psi}\cos\theta + \dot{\phi}. \end{aligned} \tag{18}$$

This can be done using $\mathbf{e}_N$ in Eq. (19) and $\mathbf{e}^{(3)}$ in Eq. (21) below.

Let us now find the complete relationship between the two set of vectors $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$ and $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ in terms of the Euler angles. The three vectors, $\mathbf{e}^{(3)}$, $\mathbf{e}_A$, and $\mathbf{e}_x$ lie in the same plane. The auxiliary vectors can be projected onto $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ as follows

$$\begin{aligned} \mathbf{e}_N &= \cos\phi\mathbf{e}_x + \sin\phi\mathbf{e}_y \\ \mathbf{e}_A &= -\cos\theta\sin\phi\mathbf{e}_x + \cos\theta\cos\phi\mathbf{e}_y + \sin\theta\mathbf{e}_z. \end{aligned} \tag{19}$$

To project $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$ onto $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$, one can at first project $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ onto $\mathbf{e}_N$ and $\mathbf{e}_A$

$$\begin{aligned} \mathbf{e}^{(1)} &= \cos\psi\mathbf{e}_N + \sin\psi\mathbf{e}_A \\ \mathbf{e}^{(2)} &= -\sin\psi\mathbf{e}_N + \cos\psi\mathbf{e}_A \end{aligned} \tag{20}$$

and then use Eq. (19). Vector $\mathbf{e}^{(3)}$ can be projected onto $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ directly. As the result one obtains

$$\begin{aligned} \mathbf{e}^{(1)} &= (\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi)\,\mathbf{e}_x + (\sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi)\,\mathbf{e}_y + \sin\theta\sin\psi\mathbf{e}_z \\ \mathbf{e}^{(2)} &= (-\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi)\,\mathbf{e}_x + (-\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi)\,\mathbf{e}_y + \sin\theta\cos\psi\mathbf{e}_z \\ \mathbf{e}^{(3)} &= \sin\theta\sin\phi\mathbf{e}_x - \sin\theta\cos\phi\mathbf{e}_y + \cos\theta\mathbf{e}_z. \end{aligned} \tag{21}$$

Coefficient in these equations form a 3×3 rotation matrix properties of which will be studied below in a more general context.

1.5 Rotation matrices

Material of this section can be skipped in the first reading.

1.5.1 General

A number of properties of the rotation matrices can be proven. In particular, these matrices are orthogonal,

$$\mathbf{A}^T \cdot \mathbf{A} \equiv \mathbf{A} \cdot \mathbf{A}^T = \mathbf{1}, \qquad \mathbf{A}^T = \mathbf{A}^{-1}, \tag{22}$$

where $\mathbf{A}^T$ is the matrix transposed to $\mathbf{A}$. Transformations by orthogonal matrices (i.e., rotations) preserve the length of the vectors, $\mathbf{r}'^2 = \mathbf{r}^2$.

1.5.2 Active and passive rotations

One has to distinguish between active and passive rotations. Active rotations are actual rotations of vectors such as the rotation described by Eq. (1). Active rotations are described with respect to the unchanged coordinate system and are represented by

$$r'_\alpha = A_{\alpha\beta} r_\beta, \qquad \mathbf{r}' = \mathbf{A} \cdot \mathbf{r} \tag{23}$$

(with summation over repeated indices), where $\mathbf{A}$ is a matrix of elements $A_{\alpha\beta}$ and $\mathbf{r}$ is a column of components r_β. Passive rotations are rotations of the frame axes whereas $\mathbf{r}$ remains unchanged in the original frame. With respect to the new rotated frame, the components of $\mathbf{r}$ are rotated, however. This *passive* rotation of $\mathbf{r}$ is in the opposite direction with respect to the active rotation of the frame vectors. Passive rotation is used to relate componens of a vector in different frames with each other.

Let $\mathbf{e}_\gamma$ be the original frame vectors, such as $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ of the laboratory system and $\mathbf{e}^{(\gamma)}$ are rotated vectors such as the body-frame vectors $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, and $\mathbf{e}^{(3)}$. In the initial state before rotation $\mathbf{e}^{(\gamma)} = \mathbf{e}_\gamma$. The rotation of the vectors $\mathbf{e}^{(\gamma)} \equiv \mathbf{e}'_\gamma$ is the active rotation defined by Eq. (23) in the vector form

$$\mathbf{e}^{(\gamma)} = \mathbf{A} \cdot \mathbf{e}_\gamma \tag{24}$$

($\mathbf{e}_\gamma$ is a column of its components) or

$$e^{(\gamma)}_\alpha = A_{\alpha\beta} \left(\mathbf{e}_\gamma\right)_\beta = A_{\alpha\beta}\delta_{\beta\gamma} = A_{\alpha\gamma}. \tag{25}$$

The whole vector $\mathbf{e}^{(\gamma)}$ is thus given by

$$\mathbf{e}^{(\gamma)} = e^{(\gamma)}_\alpha \mathbf{e}_\alpha = A_{\alpha\gamma}\mathbf{e}_\alpha = \left(\mathbf{A}^T\right)_{\gamma\alpha} \mathbf{e}_\alpha, \tag{26}$$

There is a subtle difference with Eq. (24): In the former $\mathbf{e}_\gamma$ is a single vector, the initial value of $\mathbf{e}^{(\gamma)}$, whereas in the latter there are three different vectors $\mathbf{e}_\alpha$ on which $\mathbf{e}^{(\gamma)}$ is projected. In particular, the coefficients in Eq. (21) directly form the matrix $\mathbf{A}^T$.

Let us consider passive rotations now. The vector $\mathbf{r}$ remains the same and it can be projected on both frames as follows

$$\mathbf{r} = r_\gamma \mathbf{e}_\gamma = r^{(\gamma)}\mathbf{e}^{(\gamma)}. \tag{27}$$

Using Eq. (26) one can relate the components of $\mathbf{r}$ in both frames:

$$\mathbf{r} = r^{(\gamma)}\mathbf{e}^{(\gamma)} = r^{(\gamma)} A_{\alpha\gamma}\mathbf{e}_\alpha = A_{\alpha\gamma} r^{(\gamma)}\mathbf{e}_\alpha, \tag{28}$$

that is,

$$r_\alpha = A_{\alpha\gamma} r^{(\gamma)}, \qquad r^{(\gamma)} = \left(\mathbf{A}^T\right)_{\gamma\alpha} r_\alpha, \tag{29}$$

where we used the orthogonality of $\mathbf{A}$, i.e., Eq. (22) or $\left(\mathbf{A}^T\right)_{\gamma'\alpha} A_{\alpha\gamma} = \delta_{\gamma\gamma'}$. One can see that since $r^{(\gamma)} = r'_\gamma$, the passive rotation is done by the inverse rotation matrix,

$$r'_\alpha = \left(\mathbf{A}^T\right)_{\alpha\beta} r_\beta, \qquad \mathbf{r}' = \mathbf{A}^T \cdot \mathbf{r}, \tag{30}$$

i.e., in the opposite direction compared to the active rotation, c.f. Eq. (23). Difference between active and passive rotations is a subtle point that is ignored in many publications that leads to uncertainties and errors.

1.5.3 Time-dependent rotation and angular velocity

Let us consider orientations of the body at time t as obtained from the reference orientation $\mathbf{e}^{(\gamma)} = \mathbf{e}_\gamma$ at $t = 0$ by a rotations matrix that depends on time. Then any vector $\mathbf{r}$ rotating with the body is given by

$$\mathbf{r}(t) = \mathbf{A}(t) \cdot \mathbf{r}_0, \tag{31}$$

where $\mathbf{r}_0 = \mathbf{r}(0)$ and we neglected trivial translational motion. The velocity of the point represented by $\mathbf{r}$ obeys the equation

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{A}} \cdot \mathbf{r}_0 = \mathbf{\Omega} \cdot \mathbf{r}, \tag{32}$$

where the matrix $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega} \equiv \dot{\mathbf{A}} \cdot \mathbf{A}^T. \tag{33}$$

Note that we are performing an active rotation while passive rotation is unsuitable for this purpose. The matrix $\mathbf{\Omega}$ is antisymmetric. Indeed, from Eq. (22) one obtains

$$\dot{\mathbf{A}} \cdot \mathbf{A}^T + \mathbf{A} \cdot \dot{\mathbf{A}}^T = 0 \qquad \Rightarrow \qquad \dot{\mathbf{A}} \cdot \mathbf{A}^T = -\mathbf{A} \cdot \dot{\mathbf{A}}^T. \tag{34}$$

Now transposing $\mathbf{\Omega}$ gives

$$\mathbf{\Omega}^T = \left(\dot{\mathbf{A}} \cdot \mathbf{A}^T \right)^T = \mathbf{A} \cdot \dot{\mathbf{A}}^T = -\dot{\mathbf{A}} \cdot \mathbf{A}^T = -\mathbf{\Omega}. \tag{35}$$

Thus all diagonal elements of $\mathbf{\Omega}$ are zero and non-diagonal elements are antisymmetric and can be represented by only three different numbers as follows

$$\mathbf{\Omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}, \qquad \Omega_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}\omega_\gamma, \tag{36}$$

where $\epsilon_{\alpha\beta\gamma}$ is fully antisymmetric unit tensor. Thus Eq. (32) takes the form

$$v_\alpha = -\epsilon_{\alpha\beta\gamma}\omega_\gamma r_\beta = \epsilon_{\alpha\beta\gamma}\omega_\beta r_\gamma = [\boldsymbol{\omega} \times \mathbf{r}]_\alpha \tag{37}$$

that coincides with Eq. (7). One can resolve Eq. (36) for ω_γ by multiplying it by $\epsilon_{\alpha\beta\gamma'}$ and summing over α and β,

$$\epsilon_{\alpha\beta\gamma'}\Omega_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma'}\epsilon_{\alpha\beta\gamma}\omega_\gamma = -2\omega_{\gamma'} \tag{38}$$

that yields

$$\omega_\gamma = -\frac{1}{2}\Omega_{\alpha\beta}\epsilon_{\alpha\beta\gamma}. \tag{39}$$

These are the components of the angular velocity $\boldsymbol{\omega}$ in the laboratory frame. The components of this same vector in the body frame can be obtained by the *passive* rotation. Putting the component indices up ($\omega^{(1)} \equiv \omega_1$ etc.) to avoid confusion and using the second equation in Eq. (29), one obtains

$$\omega^{(\gamma)} = \left(\mathbf{A}^T\right)_{\gamma\delta}\omega_\delta = \omega_\delta A_{\delta\gamma} = -\frac{1}{2}\Omega_{\alpha\beta}\epsilon_{\alpha\beta\delta}A_{\delta\gamma}. \tag{40}$$

The formalism in this section does not imply any particular parametrization of $\mathbf{A}$ through rotation angles. The latter can be Euler angles, the angles that specify a single rotation, or any other parameters. Examples will be considered below.

1.5.4 Euler angles

As follows from Eqs. (21) and (26),

$$\mathbf{A}^T = \begin{pmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \tag{41}$$

hence

$$\mathbf{A} = \begin{pmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & -\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}. \tag{42}$$

(H. Goldstein in his Classical Mechanics considers *passive* rotations, thus his passive rotation $\mathbf{A}$ is the same as our $\mathbf{A}^T$, while our $\mathbf{A}$ is the active rotation.) For the rotation matrix above, Eq. (36) takes the form

$$\mathbf{\Omega} = \begin{pmatrix} 0 & -\left(\dot\psi\cos\theta + \dot\phi\right) & -\dot\psi\sin\theta\cos\phi + \dot\theta\sin\phi \\ \dot\psi\cos\theta + \dot\phi & 0 & -\left(\dot\psi\sin\theta\sin\phi + \dot\theta\cos\phi\right) \\ -\left(-\dot\psi\sin\theta\cos\phi + \dot\theta\sin\phi\right) & \dot\psi\sin\theta\sin\phi + \dot\theta\cos\phi & 0 \end{pmatrix} \tag{43}$$

from which the laboratory-frame components of $\boldsymbol{\omega}$ can be read out. One can see that they coincide with those given by Eq. (18). Eq. (40) yields components of $\boldsymbol{\omega}$ in the body frame that are given by Eq. (17).

1.5.5 Single rotation

As mentioned below, Eq. (1) can be represented as acting a matrix on the vector $\mathbf{r}$. To work out the form of this matrix, we represent Eq. (1) via components with respect to the laboratory axes $\mathbf{e}_\alpha$

$$r'_\alpha = r_\alpha\cos\chi + \nu_\alpha\nu_\beta r_\beta\left(1-\cos\chi\right) + \epsilon_{\alpha\beta\gamma}\nu_\beta r_\gamma\sin\chi. \tag{44}$$

This can be rewritten in the form of Eq. (23) where the elements of the rotation matrix $\mathbf{A}$ are given by

$$A_{\alpha\beta} = \delta_{\alpha\beta}\cos\chi + \nu_\alpha\nu_\beta\left(1-\cos\chi\right) - \epsilon_{\alpha\beta\gamma}\nu_\gamma\sin\chi. \tag{45}$$

The matrix of small rotations at linear order in $\delta\chi$ becomes

$$A_{\alpha\beta} = \delta_{\alpha\beta} - \epsilon_{\alpha\beta\gamma}\delta\chi_\gamma, \qquad \delta\chi_\gamma \equiv \nu_\gamma\delta\chi. \tag{46}$$

The components of $\boldsymbol{\nu}$ are explicitly given by

$$\nu_x = \sin\vartheta\cos\varphi, \qquad \nu_y = \sin\vartheta\sin\varphi, \qquad \nu_z = \cos\vartheta. \tag{47}$$

Substitution yields the single-rotation matrix

$$\mathbf{A} = \begin{pmatrix} \cos\chi + (1-\cos\chi)\sin^2\vartheta\cos^2\varphi & (1-\cos\chi)\sin^2\vartheta\sin\varphi\cos\varphi - \sin\chi\cos\vartheta & (1-\cos\chi)\cos\vartheta\sin\vartheta\cos\varphi + \sin\chi\sin\vartheta\sin\varphi \\ (1-\cos\chi)\sin^2\vartheta\cos\varphi\sin\varphi + \sin\chi\cos\vartheta & \cos\chi + (1-\cos\chi)\sin^2\vartheta\sin^2\varphi & \cos\vartheta(1-\cos\chi)\sin\vartheta\sin\varphi - \sin\chi\cos\varphi\sin\vartheta \\ \cos\vartheta\cos\varphi(1-\cos\chi)\sin\vartheta - \sin\vartheta\sin\varphi\sin\chi & (1-\cos\chi)\sin\vartheta\cos\vartheta\sin\varphi + \sin\chi\cos\varphi\sin\vartheta & (1-\cos\chi)\cos^2\vartheta + \cos\chi \end{pmatrix} \tag{48}$$

In the special cases of rotation around the axes x, y, and z this matrix simplifies to

$$\mathbf{A}^{(x)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\chi & -\sin\chi \\ 0 & \sin\chi & \cos\chi \end{pmatrix}, \qquad \mathbf{A}^{(y)} = \begin{pmatrix} \cos\chi & 0 & \sin\chi \\ 0 & 1 & 0 \\ -\sin\chi & 0 & \cos\chi \end{pmatrix}, \qquad \mathbf{A}^{(z)} = \begin{pmatrix} \cos\chi & -\sin\chi & 0 \\ \sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{49}$$

In the general case, however, this matrix is much more cumbersome than the rotation matrix of three successive rotations by Euler angles, Eq. (42). Also components of $\boldsymbol{\omega}$ in both the laboratory and body frames have a complicated form. This is the reason why the single-rotation matrix is never used.

The matrices $\mathbf{A}(\theta, \phi, \psi)$ of Eq. (42) and $\mathbf{A}(\vartheta, \varphi, \chi)$ of Eq. (48) should coincide with a proper choice of the angles. Indeed, according to the Euler theorem the three rotations by Euler angles should be equivalent to a single rotation. This means that there is a vector collinear with the rotation axis that is invariant under rotation. This is $\boldsymbol{\nu}$ given by Eq. (47) The same vector should be the eigenvector of $\mathbf{A}(\theta, \phi, \psi)$ of Eq. (42) with eigenvalue $\lambda = 1$. Thus, if the rotation matrix $\mathbf{A}(\theta, \phi, \psi)$ is given, one can find its eigenvector of eigenvalue 1 and equate it to $\boldsymbol{\nu}$, thus obtaining the direction of the single-rotation axis in terms of the Euler angles. However, it is difficult to do analytically in the general case. On the other hand, equating the traces of both matrices one easily obtains

$$\cos\theta + (1+\cos\theta)\cos(\phi+\psi) = 1 + 2\cos\chi \tag{50}$$

that defines the single-rotation angle χ in terms of the Euler angles. Using computer algebra on Eqs. (39) and (40), one can calculate the components of $\boldsymbol{\omega}$ in the laboratory and body frames in terms of ϑ, φ, χ and their first derivatives that turn to be cumbersome. Although the parametrization of the body orientations through single rotations out of a reference state is a valid approach that could be used instead of the parametrization based on the Euler angles, at least for numerical work, it is never practically applied.

2 Basic dynamics of rotational motion

2.1 Kinetic energy of a rotating body, moments of inertia

Inserting Eq. (9) into the general expression for the kinetic energy one proceeds as

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i \mathbf{v}_i^2 = \frac{1}{2}\sum_i m_i \left(\mathbf{V} + [\boldsymbol{\omega}\times\mathbf{r}_i]\right)^2 = \sum_i m_i \left(\frac{1}{2}\mathbf{V}^2 + \mathbf{V}\cdot[\boldsymbol{\omega}\times\mathbf{r}_i] + \frac{1}{2}[\boldsymbol{\omega}\times\mathbf{r}_i]^2\right) \\ &= \frac{1}{2}M\mathbf{V}^2 + M\mathbf{V}\cdot\left[\boldsymbol{\omega}\times\tilde{\mathbf{R}}\right] + \frac{1}{2}\sum_i m_i \left\{\boldsymbol{\omega}^2\mathbf{r}_i^2 - (\boldsymbol{\omega}\cdot\mathbf{r}_i)^2\right\}. \end{aligned} \tag{51}$$

To transform the last term, the identity

$$[\mathbf{a}\times\mathbf{b}]\cdot[\mathbf{c}\times\mathbf{d}] = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{52}$$

was used. If the body rotates around a point of support or around an instantaneous axis of rotation, on in fact uses Eq. (7), so that the first and second terms in Eq. (51) do not occur. In the second term $\tilde{\mathbf{R}}$ is the position of the CM with respect to the origin O'. If the latter is put in the CM, $O' = O$, then $\tilde{\mathbf{R}} = \mathbf{0}$ and the second term is absent. In this case the first term in Eq. (51) is the kinetic energy of the translational motion of the center of mass. The last term in Eq. (51), the rotational energy, can be rewritten in components as follows:

$$T_{\mathrm{rot}} = \frac{1}{2}\sum_i m_i \left\{\omega_\alpha\omega_\beta\delta_{\alpha\beta}r_{i\gamma}^2 - \omega_\alpha r_{i\alpha}\omega_\beta r_{i\beta}\right\} = \frac{1}{2}I_{\alpha\beta}\omega_\alpha\omega_\beta, \tag{53}$$

where $I_{\alpha\beta}$ is the tensor of inertia defined by

$$I_{\alpha\beta} = \sum_i m_i \left(\delta_{\alpha\beta}r_{i\gamma}^2 - r_{i\alpha}r_{i\beta}\right). \tag{54}$$

If $I_{\alpha\beta}$ is defined with respect to the CM, the tensor of inertia $I'_{\alpha\beta}$ with respect to any other origin O' can be obtained by using $\mathbf{r} = \mathbf{a} + \mathbf{r}'$ and thus $\mathbf{r}' = \mathbf{r} - \mathbf{a}$ that yields

$$\begin{aligned} I'_{\alpha\beta} &= \sum_i m_i \left\{\delta_{\alpha\beta}r_{i\gamma}'^2 - r'_{i\alpha}r'_{i\beta}\right\} = \sum_i m_i \left\{\delta_{\alpha\beta}(r_{i\gamma} - a_\gamma)(r_{i\gamma} - a_\gamma) - (r_{i\alpha} - a_\alpha)(r_{i\beta} - a_\beta)\right\} \\ &= I_{\alpha\beta} + M\left\{\delta_{\alpha\beta}a_\gamma a_\gamma - a_\alpha a_\beta\right\}, \end{aligned} \tag{55}$$

where we used $\sum_i m_i r_{i\gamma} = 0$.

Eq. (54) works out to

$$I_{xx} = \sum_i m_i \left(y_i^2 + z_i^2 \right), \qquad I_{xy} = -\sum_i m_i x_i y_i, \tag{56}$$

etc. In the diagonal components of $I_{\alpha\beta}$ the squares of the distances from the corresponding axes enter, like $y_i^2 + z_i^2$ being the square of the distance from the x-axis. For solid bodies moments of inertia can be obtained by replacing summation by integration. In some cases symmetry can be used to simplify the calculation. Eq. (55) yields

$$I'_{xx} = I_{xx} + M \left(a_y^2 + a_z^2 \right), \tag{57}$$

etc. Here $a_y^2 + a_z^2$ is the square of the distance between the x-axis and the CM. One can see that diagonal moments of inertia are minimal with respect to the CM and the shifted moments of inertia are greater than CM-centered ones.

In Eqs. (54) and below moments of inertia are written with respect to an arbitrarily oriented coordinate system. If the laboratory coordinate system is used then, in general, components of the tensor of inertia $I_{\alpha\beta}$ change with time, as the body rotates. This is very unconvenient and makes writing down a closed system of equations of motion hardly possible. Much better choice is to use the body's own embedded coordinate system with respect to which $I_{\alpha\beta}$ are independent of the body's orientation. Moreover, for a body of any shape, the tensor of inertia can be diagonalized by an appropriate choice of the orientation of the embedded Descarte axes. Such axes are called principal axes. Then the rotational energy has the simple form

$$T_{\text{rot}} = \frac{1}{2} \left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \right). \tag{58}$$

In most practical case bodies of a symmetric shape are considered for which the choice of the internal coordinate system is obvious.

Substituting Eq. (17) into Eq. (58) gives the expression of the rotational kinetic energy in terms of Euler angles

$$T_{\text{rot}} = \frac{1}{2} I_1 \left(\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \right)^2 + \frac{1}{2} I_2 \left(\dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \right)^2 + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2. \tag{59}$$

In the case of symmetric top ($I_1 = I_2$) simplification yields

$$T_{\text{rot}} = \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos\theta + \dot{\psi} \right)^2. \tag{60}$$

In fact, for the symmetric top the orientation of the vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ is not fixed by any condition. One can even choose, instead of embedded vectors, "sliding" vectors so that $\mathbf{e}^{(1)} = \mathbf{e}_N$ for all times and $\mathbf{e}^{(2)}$ is perpendicular to it. Setting $\psi = 0$ in Eq. (16) one obtains

$$\omega_1 = \dot{\theta}, \qquad \omega_2 = \dot{\phi} \sin\theta, \qquad \omega_3 = \dot{\phi} \cos\theta + \dot{\psi}. \tag{61}$$

Inserting this into Eq. (58) yields again Eq. (60). Since T_{rot} contains terms beyond quadratic in generalized coordinates and velocities, rotational motion of even a free body can be very complicated.

2.2 Angular momentum

2.2.1 Angular momentum in laboratory and body frames

Let us now express the angular momentum $\mathbf{L}$ through the components of the angular velocity $\boldsymbol{\omega}$. Using Eqs. (9) and (2), one proceeds as

$$\mathbf{L} = \sum_i m_i \left[\mathbf{r}_i \times \mathbf{v}_i \right] = \sum_i m_i \left[\mathbf{r}_i \times \left(\mathbf{V} + \left[\boldsymbol{\omega} \times \mathbf{r}_i \right] \right) \right] = M \left[\mathbf{R} \times \mathbf{V} \right] + \sum_i m_i \left\{ \boldsymbol{\omega} \mathbf{r}_i^2 - \mathbf{r}_i \left(\boldsymbol{\omega} \cdot \mathbf{r}_i \right) \right\}. \tag{62}$$

The first term of this result is the angular momentum associated with translation motion of the center of mass. This contribution is trivial and it can be removed by choosing the frame that is moving with the CM. If rotation around a point of support is considered, Eq. (7) is used and this term does not arise. Henceforth we will consider only the last term in Eq. (62) that is due to the rotation of the body. In components it has the form

$$L_\alpha = I_{\alpha\beta}\omega_\beta, \tag{63}$$

where the tensor of inertia is given by Eq. (54). The angular momentum $\mathbf{L}$ in the absence of torques acting on the system is conserved, $\mathbf{L} = \mathbf{const}$. Thus all components L_α with respect to the laboratory coordinate system are constants. As the body rotates, $I_{\alpha\beta}$ with respect to the laboratory frame in general is changing, apart of special cases such as spherical body. Thus the vector $\boldsymbol{\omega}$ is changing, too, so that the orientation of the body changes in time in a complicated way. The relation between the components of $\mathbf{L}$ and $\boldsymbol{\omega}$ simplifies in the principal coordinate system,

$$\mathbf{L} = I_1\omega_1\mathbf{e}^{(1)} + I_2\omega_2\mathbf{e}^{(2)} + I_3\omega_3\mathbf{e}^{(3)}. \tag{64}$$

With the help of this the rotational kinetic energy of Eq. (58) can be rewritten in the form

$$T_{\text{rot}} = \frac{1}{2}\left(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right). \tag{65}$$

Comparing Eqs. (53) and (63), one can easily obtain the relations

$$\mathbf{L} = \frac{\partial T_{\text{rot}}}{\partial \boldsymbol{\omega}}, \qquad L_\alpha = \frac{\partial T_{\text{rot}}}{\partial \omega_\alpha}. \tag{66}$$

This means that the angular momentum plays a role of a generalized momentum within the Lagrangian formalism.

2.2.2 Equation of motion for the angular momentum

The angular momentum $\mathbf{L}$ introduced in Sec. 2.2 obeys the equation of motion that can be obtained from the Newton's second law $m\dot{\mathbf{v}} = \mathbf{f}$ as follows

$$\dot{\mathbf{L}} = \frac{d}{dt}\sum_i m_i\left[\mathbf{r}_i \times \mathbf{v}_i\right] = \sum_i m_i\left\{\underbrace{\left[\dot{\mathbf{r}}_i \times \mathbf{v}_i\right]}_{0} + \left[\mathbf{r}_i \times \dot{\mathbf{v}}_i\right]\right\} = \sum_i \left[\mathbf{r}_i \times \mathbf{f}_i\right] = \mathbf{K}, \tag{67}$$

where $\mathbf{K}$ is the total torque acting on the system. In the expression for $\mathbf{K}$ internal forces are cancelled according to the Newton's third law so that only the external forces remain. The torque depends, as also $\mathbf{L}$, on the choice of the origin of the coordinate system. Usual choices of the origin are the center of mass, the point of support, or an instantaneous axis of rotation. It is easy to show that if the total force acting on a system is zero, $\mathbf{F} = 0$, the torque is independent of the origin placement. For $\mathbf{K} = \mathbf{0}$ the angular momentum is an integral of motion, $\mathbf{L} = \mathbf{const}$, that was already used in Sec. 2.2.4. A torque created by potential forces is related to the dependence of the potential energy U on the body's orientation. Infinitesimal change of U due to rotations can be represented as

$$\delta U = -\sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = -\sum_i \mathbf{f}_i \cdot \left[\delta\boldsymbol{\chi} \times \mathbf{r}_i\right] = -\sum_i \delta\boldsymbol{\chi} \cdot \left[\mathbf{r}_i \times \mathbf{f}_i\right] = -\mathbf{K}\cdot\delta\boldsymbol{\chi}, \tag{68}$$

thus

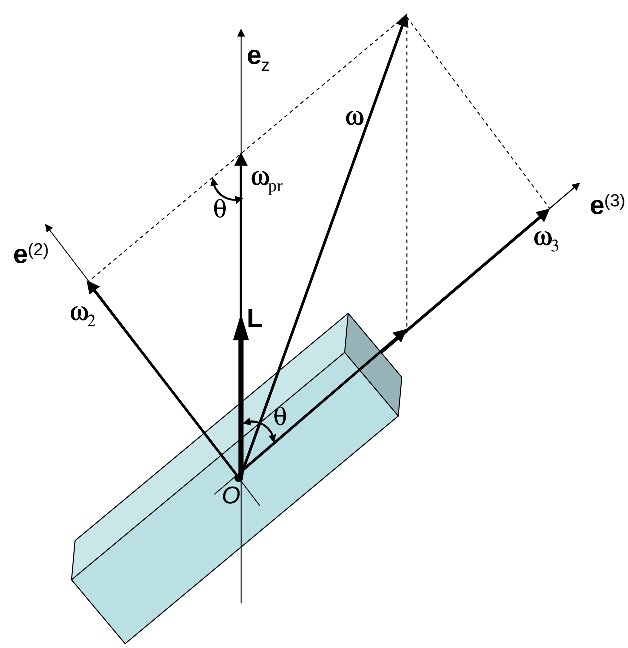
$$\mathbf{K} = -\frac{\delta U}{\delta \boldsymbol{\chi}}. \tag{69}$$

Here we avoid writing $\mathbf{K} = -\partial U(\boldsymbol{\chi})/\partial\boldsymbol{\chi}$ since the function $U(\boldsymbol{\chi})$ that could be differentiated does not exist. On the other hand, using Euler angles one obtains the potential energy as $U = U(\theta, \phi, \psi)$ that can be differentiated. In particular, one obtains

$$K_N = -\frac{\partial U}{\partial \theta}, \qquad K_z = -\frac{\partial U}{\partial \phi}, \qquad K_3 = -\frac{\partial U}{\partial \psi}, \tag{70}$$

where K_N is projection of the torque on the line of nodes.

2.2.3 Free symmetric top



The tensor relation between $\mathbf{L}$ and $\boldsymbol{\omega}$, Eq. (63) makes the rotational dynamics complicated. To illustrate this, consider a free symmetric top $I_1 = I_2 \neq I_3$ having the angular momentum $\mathbf{L}$. For a free body $\mathbf{L} = \mathbf{const}$, thus it is convenient to choose $\mathbf{e}_z$ along $\mathbf{L}$. In general, $\mathbf{L}$ does not coincide with the principal axes $\mathbf{e}^{(\alpha)}$. Thus $\boldsymbol{\omega}$ is not collinear to $\mathbf{L}$. The body-frame components of $\mathbf{L}$ and $\boldsymbol{\omega}$ are given by

$$\begin{aligned} L_1 &= 0, \qquad L_2 = L\sin\theta, \qquad L_3 = L\cos\theta \\ \omega_1 &= 0, \qquad \omega_2 = \frac{L}{I_2}\sin\theta, \qquad \omega_3 = \frac{L}{I_3}\cos\theta, \end{aligned} \tag{71}$$

where we have chosen $\mathbf{e}^{(1)} \perp \mathbf{e}_z$ using the freedom of choice of $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ for an axially symmetric body. One can see that $\boldsymbol{\omega}$ is in the same plane as $\mathbf{L}$ and $\mathbf{e}^{(3)}$. The velocity of any point $\mathbf{r} = r_3\mathbf{e}^{(3)}$ on the symmetry axis

$$\mathbf{v} = [\boldsymbol{\omega} \times \mathbf{r}] = r_3\left[\boldsymbol{\omega} \times \mathbf{e}^{(3)}\right] = r_3\omega_2\left[\mathbf{e}^{(2)} \times \mathbf{e}^{(3)}\right] = r_3\omega_2\mathbf{e}^{(1)} \tag{72}$$

is perpendicular to $\mathbf{e}_z$ and $\mathbf{e}^{(3)}$. Thus the symmetry axis $\mathbf{e}^{(3)}$ is precessing around $\mathbf{e}_z$. Since $\boldsymbol{\omega}$ lies in the plane specified by $\mathbf{e}_z$ and $\mathbf{e}^{(3)}$, it is precessing at the same rate. The rate of precession $\omega_{\text{pr}} = \dot{\phi}$ can be found by dividing v by the distance from the axis z:

$$\dot{\phi} = \frac{v}{r_3\sin\theta} = \frac{\omega_2}{\sin\theta} = \frac{L}{I_2}. \tag{73}$$

Alternatively, one can resolve the vector $\boldsymbol{\omega}$ in components along $\mathbf{e}^{(3)}$ and $\mathbf{e}_z$, the latter being $\omega_{\rm pr}$ (see figure). One obtains $\omega_{\rm pr} = \omega_2/\sin\theta$ as above.

Using the last of Eqs. (71) and the last of Eqs. (17) one obtains

$$\dot{\psi} = \left(\frac{1}{I_3} - \frac{1}{I_1}\right) L\cos\theta. \tag{74}$$

This formua can be rewritten as

$$\dot{\psi} = \left(1 - \frac{I_3}{I_1}\right)\omega_3 \tag{75}$$

that nicely illustrates that $\dot{\psi}$ and ω_3 are not the same. One can also relate the precession (wobble) and spin using Eq. (73):

$$\dot{\psi} = \left(\frac{I_1}{I_3} - 1\right)\dot{\phi}\cos\theta. \tag{76}$$

The Earth is very close to a fully symmetric top,

$$1 - \frac{I_3}{I_1} \simeq -\frac{1}{300}, \tag{77}$$

and the angle θ is small. The small deviation from the full symmetry must be due to the centrifugal forces caused by the Earth's rotation and the Earth's incomplete rigidity. Thus the apparent rate of rotation of the Earth ω_3 that leads to the change of day and night in fact is mostly precession $\dot{\phi}$ of the Earth's symmetry axis (wobble), $\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \cong \dot{\phi}$. The actual Earth's spin $\dot{\psi}$ is very small, and the period of this rotation should be 300 days, according to Eqs. (75) and (77). If $\dot{\psi}$ were zero, the vector $\boldsymbol{\omega}$ would cross the Earth's surface at a fixed point (about 10 m from the North Pole). Because of the rotation $\dot{\psi}$, the crossing point should be non-stationary, making circles around the North Pole with a period of 300 days. Observations show that a similar effect exists, and it is called Chandler wobbling. However, the period of the Chandler wobbling is about 427 days and the trajectory of the crossing point has a large irregular component. The latter can be blamed on the Earth's incomplete rigidity, whereas the theoretical result 300 days was obtained for a rigid body (already by Leonhard Euler). The fact that the Earth's rate of rotation around its symmetry axis is very small seems to contradict our everyday experience. However, since θ is small and thus the axes $\mathbf{e}_z$ and $\mathbf{e}^{(3)}$ are close to each other, it makes little difference for an observer whether the body rotates around $\mathbf{e}_z$ (rate $\dot{\phi}$) or around $\mathbf{e}^{(3)}$ (rate $\dot{\psi}$).

Another example is a disc that rotates freely aroud the axis nearly parallel to the symmetry axis, $\theta \ll 1$. Since for a disc $I_3 = 2I_1$, the relation between the wobble and the spin, Eq. (76), becomes

$$\dot{\phi} \cong -2\dot{\psi}. \tag{78}$$

If $I_3 \to 0$ like for an infinitely thin rod, Eq. (76) yields $\dot{\psi}$ diverging as $\dot{\psi} \cong (I_1/I_3)\,\dot{\phi}\cos\theta$. According to Eq. (17) this yields the diverging $\omega_3 \cong \dot{\psi} \cong (I_1/I_3)\,\dot{\phi}\cos\theta$ and further the diverging kinetic energy $E \cong (1/2)\,I_3\omega_3^2 \cong (1/2)\left(I_1^2/I_3\right)\dot{\phi}^2\cos^2\theta$. For $\dot{\phi} \neq 0$, the only chance to avoid this divergence is to have $\theta = \pi/2$. The same can be seen on a more basic level. The kinetic-energy contribution $L_3^2/(2I_3)$ in Eq. (65) should not diverge for $I_3 \to 0$, thus L_3 must vanish. According to Eq. (84) it requires $\cos\theta \to 0$. Thus a rod-like body with $I_3 \to 0$ would freely rotate with $\mathbf{L}$ perpendicular to its length. Deviations from this orientation would induce very large $\dot{\psi}$ that cost much kinetic energy.

Let us now consider the dynamics of $\boldsymbol{\omega}$ in the laboratory frame in more detail. As said above, it is precessing at the rate $\dot{\phi}$ around the constant vector $\mathbf{L}$. The solution can be written down from geometric arguments. Als one can use equations (18) with $\dot{\theta} = 0$ that yield

$$\begin{aligned}
\omega_x &= \dot{\psi}\sin\theta\sin\phi \\
\omega_y &= -\dot{\psi}\sin\theta\cos\phi \\
\omega_z &= \dot{\psi}\cos\theta + \dot{\phi}.
\end{aligned} \tag{79}$$

With the help of Eq. (86) one obtains

$$\omega_z = \frac{L}{I_1} + \left(\frac{I_1}{I_3} - 1\right)\frac{L}{I_3}\cos^2\theta = \mathrm{const} \tag{80}$$

and

$$\left\{\begin{array}{c}\omega_x \\ \omega_y\end{array}\right\} = \left(\frac{I_1}{I_3} - 1\right)\frac{L}{I_3}\cos\theta\sin\theta\left\{\begin{array}{c}\sin\phi \\ -\cos\phi\end{array}\right\} \tag{81}$$

and the angle ϕ depends on time as $\phi(t) = (L/I_1)\,t + \mathrm{const}$. Ed. (81) describes precession of the vector $\boldsymbol{\omega}$ around the constant vector $\mathbf{L}$. For a nearly fully symmetric top such as the Earth this precession has a very small amplitude, although it is not slow. For $\cos\theta = 0$ or $\sin\theta = 0$ the vectors $\boldsymbol{\omega}$ and $\mathbf{L}$ are collinear and there is no precession.

2.2.4 Free asymmetric top

From Eqs. (64) and (17) one obtains

$$\begin{aligned} L_1 &= I_1\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right) \\ L_2 &= I_2\left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right) \\ L_3 &= I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right). \end{aligned} \tag{82}$$

Let us now express L_1, L_2, and L_3 through the laboratory components L_x, L_y, and L_z. Using the second formula of Eq. (29) for a passive rotation of a vector and Eq. (41), one obtains

$$\begin{aligned} L_1 &= (\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi)\,L_x + (\sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi)\,L_y + \sin\theta\sin\psi L_z \\ L_2 &= (-\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi)\,L_x + (-\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi)\,L_y + \sin\theta\cos\psi L_z \\ L_3 &= \sin\theta\sin\phi L_x - \sin\theta\cos\phi L_y + \cos\theta L_z. \end{aligned} \tag{83}$$

In the absence of torques all three components of $\mathbf{L}$ are constants, although the Euler angles are changing in time. In this case Eq. (83) together with Eq. (82) forms a system of three first-order nonlinear differential equations for $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$ with constant L_x, L_y, and L_z. One can choose $\mathbf{e}_z$ along $\mathbf{L} = \mathbf{const}$, then $L_x = L_y = 0$ and $L_z = L$. This results in the important relations

$$L_1 = L\sin\theta\sin\psi, \qquad L_2 = L\sin\theta\cos\psi, \qquad L_3 = L\cos\theta \tag{84}$$

that can be obtained directly. Substituting Eq. (82) and making linear combinations of different equations one arrives at the equations of motion for the Euler angles

$$\begin{aligned} \dot{\theta} &= \left(\frac{1}{I_1} - \frac{1}{I_2}\right) L\sin\theta\sin\psi\cos\psi \\ \dot{\phi} &= \left(\frac{\sin^2\psi}{I_1} + \frac{\cos^2\psi}{I_2}\right) L \\ \dot{\psi} &= \left(\frac{1}{I_3} - \frac{\sin^2\psi}{I_1} - \frac{\cos^2\psi}{I_2}\right) L\cos\theta. \end{aligned} \tag{85}$$

Note that equations for $\dot{\theta}$ (nutation) and $\dot{\psi}$ (spin) form an autonomous system of equations that can be solved as the first step. After that the equation for the precession $\dot{\phi}$ can be integrated using previously found $\psi(t)$. It is difficult to believe that something can be unknown in mechanics of rigid bodies. Still, this very important and beautiful system of equations is not included in Mechanics textbooks, and I was also unable to find it in a more specialized literature that I have scanned. Thus let us call Eq. (85) Garanin equations, until somebody claims the priority. For a free symmetric top, $I_1 = I_2 \neq I_3$ these equations simplify to

$$\begin{aligned} \dot{\theta} &= 0 \\ \dot{\phi} &= \frac{L}{I_1} \\ \dot{\psi} &= \left(\frac{1}{I_3} - \frac{1}{I_1}\right) L\cos\theta \end{aligned} \tag{86}$$

that coincides with the solution found in Sec. 2.2.3.

There is a big contrast with the translational dynamics of a free body for which the conservation of the momentum $\mathbf{P}$ yields the three component equations

$$\dot{r}_\alpha = P_\alpha/M, \qquad \alpha = x, y, z \tag{87}$$

that can be easily integrated. Eq. (85) is the rotational analog of the trivial equations above. However, equations for Euler angles are much more complicated because of the intrinsic nonlinearity and coupling of different rotational variables.

One can substitute Eq. (84) into the rotational kinetic energy of Eq. (65) obtaining

$$T_{\mathrm{rot}} = E = \frac{L^2}{2}\left[\frac{\sin^2\theta\sin^2\psi}{I_1} + \frac{\sin^2\theta\cos^2\psi}{I_2} + \frac{\cos^2\theta}{I_3}\right] = \mathrm{const} \tag{88}$$

that relates θ and ψ under conservation of energy and angular momentum. Note that the energy does not depend on ϕ since rotation of the system around $\mathbf{L}\|\mathbf{e}_z$ amounts to an irrelevant redefinition of $\mathbf{e}_x$ and $\mathbf{e}_y$. Expressing ψ via θ from Eq. (88) and substituting the result into the first line of Eq. (85) one obtains an isolated equation for $\dot{\theta}$ that can be solved in terms of Jacobian elliptic functions. In a similar way one can obtain an isolated equation for $\dot{\psi}$. Much easier, however, is to solve Eq. (85) numerically. Application of the above equations to the asymmetric top will be discussed below.

At the end of this section, we give the formula for L_z in terms of Euler angles that will be needed below. The first formula of Eq. (29) for a passive rotation of a vector and Eq. (42) yield

$$L_z = \sin\theta\sin\psi L_1 + \sin\theta\cos\psi L_2 + \cos\theta L_3. \tag{89}$$

Now using Eqs. (64) and (17) one obtains

$$L_z = I_1\sin\theta\sin\psi\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right) + I_2\sin\theta\cos\psi\left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right) + I_3\cos\theta\left(\dot{\phi}\cos\theta + \dot{\psi}\right). \tag{90}$$

Here we do not assume conservation of $\mathbf{L}$. For a symmetric top, $I_1 = I_2$, this simplifies to

$$L_z = I_2\dot{\phi}\sin^2\theta + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\cos\theta. \tag{91}$$

The same result follows in the case of sliding vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ with $\psi = 0$ for a general top (see the problem wheel on a plane).

2.2.5 Larmor equation

Let us consider a body rapidly rotating around one of its principal axes, say, the 3-axis, and having a fixed point of support O' on this axis at the distance a from the center of mass O. We direct the vector $\mathbf{e}^{(3)}$ from O' to O. With respect to O' the gravity force $\mathbf{F} = -Mg\mathbf{e}_z$ produces a torque

$$\mathbf{K} = \left[a\mathbf{e}^{(3)} \times (-Mg\mathbf{e}_z)\right] = Mga\left[\mathbf{e}_z \times \mathbf{e}^{(3)}\right]. \tag{92}$$

The full energy

$$E = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) + Mga\left(\mathbf{e}_z \cdot \mathbf{e}^{(3)}\right), \tag{93}$$

see Eq. (58), is conserved. Due to the gravity, the scalar product $\left(\mathbf{e}_z \cdot \mathbf{e}^{(3)}\right)$ can change and the top initially rotating around the 3-axis can acquire rotations around the 1- and 2-axes as well. However, the energy associated with these additional rotations is only a small correction to the main rotational energy $(1/2)\, I_3\omega_3^2$ under the condition of rapid rotation $T_{\text{rot}} \gg \Delta U$ or $I_3\omega_3^2 \gg Mga$. Thus one can expect that the body will continue to rapidly rotate around the 3-axis, that is, to a good approximation,

$$\mathbf{L} = I_3\omega_3\mathbf{e}^{(3)} = L\mathbf{e}^{(3)}. \tag{94}$$

Then, combining Eqs. (67), (92), and (94), one obtains the equation of motion for $\mathbf{L}$

$$\dot{\mathbf{L}} = [\mathbf{L} \times \mathbf{\Omega}], \qquad \mathbf{\Omega} \equiv -\frac{Mga}{L}\mathbf{e}_z. \tag{95}$$

This is the famous Larmor equation describing precession of $\mathbf{L}$ around the vertical axis with the Larmor frequency $\Omega = Mga/L$. Similar equation can be written for $\mathbf{e}^{(3)}$ that describes the orientation of the top:

$$\dot{\mathbf{e}}^{(3)} = \left[\mathbf{e}^{(3)} \times \mathbf{\Omega}\right]. \tag{96}$$

Note that this precession is in the positive direction, $\dot{\phi} = \Omega > 0$, and it does not change the potential energy of the body

$$U = Mga\left(\mathbf{e}^{(3)} \cdot \mathbf{e}_z\right) = -\left(\mathbf{L} \cdot \mathbf{\Omega}\right). \tag{97}$$

$\mathbf{\Omega}$ can be interpreted as a generalized force corresponding to the variable $\mathbf{L}$:

$$\mathbf{\Omega} = -\partial U/\partial \mathbf{L}. \tag{98}$$

For this particular problem a more natural choice of $\mathbf{e}_z$ would be down in the direction of the gravity force. (The similar is done in magnetism directing $\mathbf{e}_z$ parallel to the magnetic field.) With this choice, in the formulas above one should replace $g \Rightarrow -g$ that leads to $\mathbf{\Omega} = (Mga/L)\,\mathbf{e}_z$.

One can see that for a rapidly rotating top the Larmor frequency is small. The ratio

$$\frac{\Omega}{\omega_3} = \frac{Mga}{I_3\omega_3^2} \ll 1, \tag{99}$$

according to the applicability condition stated above. This means that angular velocities of the rotations around principal axes other than the 3-axis are indeed small, so that our method is self-consistent. Eq. (95) also can be obtained from the general solution of the problem of a symmetric top, $I_1 = I_2$, in the limit of fast rotation around the 3-axis. From the theory of asymmetrical top if follows that the precession described by Eq. (95) requires $I_3 > I_1', I_2'$ or $I_3 < I_1', I_2'$. In the case of rotation around the axis with the intermediate moment of inertia, say, $I_1' < I_3 < I_2'$, it can be shown that the simple precessional solution above is unstable and that initially small ω_1 and ω_2 exponentially grow with time and become comparable with ω_3, so that the initial assumption becomes invalid.

3 Lagrangian and Newtonian formalisms

3.1 Lagrangian formalism

3.1.1 General scheme

Lagrangian formalism for rotational dynamics is most efficient in the case of holonomic constraints. For nonholonomic constraints, the more physically appealing Newtonian formalism is preferred (see below).

To set up the Lagrangian formalism, one has to express the kinetic and potential energy in terms of the generalized coordinates that in this case are the Euler angles θ, φ, and ψ (see below). The first step is to express the kinetic energy via $\mathbf{V}$ and $\boldsymbol{\omega}$ in Eq. (9).

Expressing the potential energy in terms of the Euler angles usually poses no problem. If the body has no point of support, translational and rotationaly motion separate in most practical cases. If the body has a point of support, one can consider rotations around this point without any translational motion. The Lagrangian describing the rotational motion can be written as

$$\mathcal{L} = T_{\mathrm{rot}}(\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}) - U(\theta, \phi, \psi),$$

and the Lagrange equations read

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi}, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{\partial \mathcal{L}}{\partial \psi}. \tag{100}$$

Let us at first consider the generalized momenta

$$p_\theta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\theta}}, \qquad p_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \qquad p_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}}. \tag{101}$$

Comparizon with Eq. (66) suggests that they must be related to the components of the angular momentum $\mathbf{L}$. Indeed, as ϕ is responsible for rotations around the z-axis and ψ is responsible for rotations around the 3-axis, it should be

$$p_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = L_z, \qquad p_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = L_3. \tag{102}$$

In particular, if ϕ and ψ are cyclic, $p_\phi = L_z$ and $p_\psi = L_3$ are integrals of motion, and we know that integrals of motion following from rotational symmetry are components of $\mathbf{L}$. One can prove Eq. (102) by using Eq. (59) for the rotational energy. Then L_3 can be identified using the third line of Eq. (82) and L_z can be identified using Eq. (90). On the other hand, p_θ is the projection of $\mathbf{L}$ on $\mathbf{e}_N$, thus it is a combination of different components of $\mathbf{L}$.

3.1.2 Free symmetric top revisited

Consider, as an illustration, a free symmetric top, $\mathcal{L} = T_{\mathrm{rot}}$ of Eq. (60). This problem was solved by two different methods above. Nevertheless, it makes sense to give it another consideration within the Largangian formalism, because the same method can be applied to the nontrivial problem of a heavy symmetric top. The variables φ and ψ are cyclic, so that one obtains

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= I_1 \dot{\phi} \sin^2\theta + I_3 \left(\dot{\phi}\cos\theta + \dot{\psi}\right)\cos\theta = L_z = \mathrm{const} \\ \frac{\partial \mathcal{L}}{\partial \dot{\psi}} &= I_3 \left(\dot{\phi}\cos\theta + \dot{\psi}\right) = L_3 = \mathrm{const}. \end{aligned} \tag{103}$$

Using these results, one can express $\dot{\phi}$ and $\dot{\psi}$ through the constants of motion,

$$\begin{aligned} \dot{\phi} &= \frac{L_z - L_3\cos\theta}{I_1 \sin^2\theta} \\ \dot{\psi} &= \frac{L_3}{I_3} - \frac{L_z - L_3\cos\theta}{I_1\sin^2\theta}\cos\theta. \end{aligned} \tag{104}$$

The Lagrange equation for θ has the form

$$\dot{p}_\theta = I_1\ddot{\theta} = I_1\dot{\phi}^2 \sin\theta\cos\theta - I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\dot{\phi}\sin\theta. \tag{105}$$

Here $\dot{\phi}$ and $\dot{\psi}$ can be eliminated using Eq. (104):

$$I_1\ddot{\theta} = \left(I_1\dot{\phi}\cos\theta - L_3\right)\dot{\phi}\sin\theta = \left(\frac{L_z - L_3\cos\theta}{\sin^2\theta}\cos\theta - L_3\right)\frac{L_z - L_3\cos\theta}{I_1\sin\theta} = \frac{(L_z\cos\theta - L_3)(L_z - L_3\cos\theta)}{I_1\sin^3\theta}. \tag{106}$$

This equation can be written in the form

$$I_1\ddot{\theta} = -\frac{dU_{\text{eff}}(\theta)}{d\theta}, \tag{107}$$

where $U_{\text{eff}}(\theta)$ is an effective potential energy for θ. Eq. (107) has an integral of motion

$$\frac{1}{2}I_1\dot{\theta}^2 + U_{\text{eff}}(\theta) = \text{const}. \tag{108}$$

$U_{\text{eff}}(\theta)$ can be found from Eq. (106) or from the energy conservation $T_{\text{rot}} = E = \text{const}$ using Eqs. (60) and (104). The result has the form

$$U_{\text{eff}}(\theta) = \frac{(L_z - L_3\cos\theta)^2}{2I_1\sin^2\theta}, \tag{109}$$

up to an irrelevant constant. Now the time dependence of θ can be found from Eq. (108), then time dependences of ϕ and ψ can be found from Eq. (104). This procedure seems to be complicated and it does not promise a simple analytical solution.

On the other hand, for a free top all components of $\mathbf{L}$ in the laboratory frame are constant, as we have seen before. One can write down the expressions for L_x and L_y using the passive transformation matrix between the body and laboratory frames, calculate their time derivatives and show with the help of the Lagrange equations that $\dot{L}_x = \dot{L}_y = 0$. After that one can choose the z axis along $\mathbf{L}$. This yields $L_z = L$ and, according to Eq. (84), $L_3 = L\cos\theta$. Now Eq. (104) simplifies to the second and third lines of Eq. (86). For θ from Eq. (106) one obtains $\ddot{\theta} = 0$ and thus $\dot{\theta} = \text{const}$. This constant should be zero since nonzero $\dot{\theta}$ would create a component of the angular momentum perpendicular to the z axis that contradicts the choice of the z axis made above. Thus all previously obtained results for a free symmetric top are reproduced.

The solution outlined above is methodologically unsatisfactory since one has to abandon the Lagrangian formalism and use other methods. L_x and L_y are hidden integrals of motion for a free symmetric top and they do not emerge in a natural way. Without the physical idea of conservation of the whole vector $\mathbf{L}$, one would not perform the check of $\dot{L}_x = \dot{L}_y = 0$, would not redefined the laboratory axes, and were satisfied with the two integrals of motion L_z and L_3 and the ensuing complicated solution mentioned after Eq. (109). On the other hand, for a heavy symmetric top the Largangian formalism is adequate since L_x and L_y are not conserved and the laboratory axes do not have to be redefined.

3.1.3 Heavy symmetric top

For a heavy symmetric top with a point of support O' on the symmetry axis at the distance a from the CM at O, one has to replace $I_1 \Rightarrow I_1' = I_1 + Ma^2$ and add the potential of gravity to the effective potential of Eq. (109). This yields

$$U_{\text{eff}}(\theta) = \frac{(L_z - L_3\cos\theta)^2}{2I_1'\sin^2\theta} + Mga\cos\theta, \tag{110}$$

whereas Eq. (104) remains the same with the replacement $I_1 \Rightarrow I_1'$. In this case L_x and L_y do not conserve anymore and the motion is more complicated. The vector $\mathbf{e}^{(3)}$ is still precessing around $\mathbf{L}$ but $\mathbf{L} \neq \mathbf{const}$ and is precessing around the vertical axis. These both precessions are not simple precessions with a constant projection angle and precession rate. One speaks about change of ϕ as *precession* and change of θ as *nutation*. If the *average* rate of precession is smaller than the rate of nutation (weak gravity), vector $\mathbf{e}^{(3)}$ makes loops and its trajectory crosses itself repeatedly. In the other case (strong gravity) the trajectory of $\mathbf{e}^{(3)}$ does not cross itself. In this case $\dot{\phi}$ given by Eq. (104) does not change sign.

Nutations are described by Eq. (108) with $U_{\text{eff}}(\theta)$ above that has upward curvature as function of $\cos\theta$. Indeed, $(L_z - L_3\cos\theta)^2$ is a parabola with the upward curvature and $1/\sin^2\theta = 1/\left(1-\cos^2\theta\right)$ has upward curvature and diverges at the boundaries of the interval, $\cos\theta = \pm 1$. The potential energy is a straight line on $\cos\theta$. Thus $U_{\text{eff}}(\theta)$ has one minimum in the interval $-1 \le \cos\theta \le 1$.

Let us consider some aspects of the motion of a heavy symmetric top.

If the top is oriented vertically up and rotating around z axis (the so-called "sleeping top"), then $\theta = 0$ and $L_3 = L_z = L$, see Eq. (71). Analyzing $U_{\text{eff}}(\theta)$ of Eq. (110) allows to find the stability region of this state. Expanding $U_{\text{eff}}(\theta)$ up to θ^2 one obtains

$$U_{\text{eff}}(\theta) \cong \left(\frac{L^2}{4I_1'} - Mga\right)\frac{\theta^2}{2}. \tag{111}$$

Thus the up state is stable if the rotation is fast enough,

$$L^2 > 4I_1' Mga. \tag{112}$$

Similar analysis shows that the rotation with the downward orientation, $L_3 = -L_z$ and $\theta = \pi$, is always stable. In all other cases $L_3 \neq \pm L_z$ and $U_{\text{eff}}(\theta)$ diverges for $\theta \to 0, \pi$. Thus $U_{\text{eff}}(\theta)$ has a single minimum inside the interval $\theta \in (0, \pi)$.

Let us find the condition for the top to move with $\theta = \text{const}$. Obviously this requires that $\theta = \theta_{\text{min}}$ corresponding to the minimum of $U_{\text{eff}}(\theta)$, that is, $U_{\text{eff}}(\theta)/d\theta = 0$. Calculating the derivative of Eq. (110) and then expressing $L_z - L_3\cos\theta$ via $\dot\phi$ and L_3 via $\dot\psi$ using Eq. (104), one obtains the stationarity condition for θ in the form

$$\dot\psi = \left(\frac{I_1'}{I_3} - 1\right)\dot\phi\cos\theta + \frac{Mga}{I_3\dot\phi}. \tag{113}$$

For a free top, $Mga = 0$, this reduces to Eq. (76) with $I_1 \Rightarrow I_1'$. If the precession rate $\dot\phi$ is small, the second term in Eq. (113) dominates, the first term can be neglected, and the relation between $\dot\phi$ and $\dot\psi$ takes the form

$$\dot\phi = \frac{Mga}{I_3\dot\psi}. \tag{114}$$

This slow-precession case is realized if the top is rapidly rotating around its symmetry axis. In this case $\mathbf{L} \cong L\mathbf{e}^{(3)}$ and $L \cong I_3\dot\psi$, so that the rate of precession $\dot\phi$ is the Larmor frequency $\Omega = Mga/L$, see Eq. (95).

One can ask what happens with a rapidly rotating symmetric top if it is not prepared in the state satisfying Eq. (113) and thus $\dot\theta = 0$ is not fulfilled. Will the Larmor equation (95) still be valid? The answer is positive. To see it, one can notice again that $\mathbf{L} \cong L_3\mathbf{e}^{(3)}$ and thus L_z can be parametrized as $L_z = L_3\cos\theta_0$ (if the top is spinning slowly, L_3 is small and such parametrization is impossible). Now Eq. (110) takes the form

$$U_{\text{eff}}(\theta) = \frac{L_3^2\left(\cos\theta_0 - \cos\theta\right)^2}{2I_1'\sin^2\theta} + Mga\cos\theta, \tag{115}$$

One can see that for a large L_3 the effective energy $U_{\text{eff}}(\theta)$ has a very deep and narrow minimum near θ_0 that is a little bit shifted towards the actual minimum at θ_{min} because of the gravity term. The motion is thus confined to a narrow region of θ in the vicinity of θ_{min}, so that the top performs fast nutations of a small amplitude around θ_{min}. Near the minimum $U_{\text{eff}}(\theta)$ can be approximated by a parabola and thus it is symmetric. As a result, the top spends equal times at $\theta < \theta_{\text{min}}$ and $\theta > \theta_{\text{min}}$, so that the average over fast nutations is $\langle\theta\rangle = \theta_{\text{min}}$. This means that on average the top moves as though $\theta = \theta_{\text{min}}$, the case considered just above. This justifies the more intuitive derivation of the Larmor equation in Sec. 2.2.2

Let us now consider the spinless regime $\dot\psi = 0$ with $\dot\theta = 0$. According to Eq. (113), this regime is realized if

$$\dot\phi = \pm\sqrt{\frac{Mga}{\left(I_3 - I_1'\right)\cos\theta}}, \qquad \left(I_3 - I_1'\right)\cos\theta > 0. \tag{116}$$

If I_3 is small, then Eq. (113) yields $\dot{\psi} \cong \left(I_1' \dot{\phi} \cos\theta + Mga/\dot{\phi} \right) / I_3$. To avoid divergence of the kinetic energy, Eq. (65), for $I_3 \to 0$, the expression in brackets must be zero and precession rate $\dot{\phi}$ must be given by

$$\dot{\phi} = \pm \sqrt{\frac{Mga}{-I_1' \cos\theta}}, \qquad \pi/2 < \theta < \pi. \tag{117}$$

This looks like a particular case of the preceding formula, although its origin is different. Eq. (117) describes stationary precession of a thin-rod pendulum that does not rotate around its symmetry axis and is fully described by θ and ϕ. Of course, for the pendulum Eq. (117) can be obtained in a shorter way.

Results for $I_3 \to 0$ can be obtained in a more general way. According to Eq. (65), for $I_3 \to 0$ must be $L_3 \to 0$, to avoid divergence of the kinetic energy. Thus one should drop L_3 in Eq. (110) and in the first line of Eq. (104). This yields the expressions for the pendulum

$$U_{\text{eff}}(\theta) = \frac{L_z^2}{2I_1' \sin^2\theta} + Mga\cos\theta, \qquad \dot{\phi} = \frac{L_z}{I_1 \sin^2\theta}, \tag{118}$$

whereas $\dot{\psi}$ becomes irrelevant and should be discarded. From here the requirement $\theta = \text{const}$ (i.e., $\partial U_{\text{eff}}(\theta)/\partial\theta = 0$) leads to Eq. (117). Note that for the pendulum the reference orientation is conventionally chosen down, so that in standard notations one should replace $\theta \to \pi - \theta$.

3.2 Newtonian formalism

Whereas in many important cases Lagrangian formalism for rotational dynamics is sufficient and leads to the result in the shortest way (such as for a cone rolling on an inclined plane without slipping), the Newtonian formalism remains the most versatile. It allows to consider systems with non-holonomic constraints in a more physically appealing way, it can be easily extended to include forces of new kinds.

3.2.1 Euler equations

Eq. (67) should be used together with Eq. (63) that yields the angular acceleration $\dot{\boldsymbol{\omega}}$ in the simplest case of rotation around a symmetry axis. However, in general vectors $\mathbf{L}$ and $\boldsymbol{\omega}$ are noncollinear and depend on time in different ways. In the laboratory frame $I_{\alpha\beta}$ changes with time as the body rotates that makes the laboratory frame unconvenient. As the simplest relation between $\mathbf{L}$ and $\boldsymbol{\omega}$ takes place in the body frame using the principal-axes, Eq. (64), changing to the body frame is inevitable. Differentiating Eq. (64) over time and taking into account that both L_α and $\mathbf{e}^{(\alpha)}$ depend on time, one obtains

$$\begin{aligned} \dot{\mathbf{L}} &= \dot{L}_1 \mathbf{e}^{(1)} + \dot{L}_2 \mathbf{e}^{(2)} + \dot{L}_3 \mathbf{e}^{(3)} + L_1 \dot{\mathbf{e}}^{(1)} + L_2 \dot{\mathbf{e}}^{(2)} + L_3 \dot{\mathbf{e}}^{(3)} \\ &= \dot{L}_1 \mathbf{e}^{(1)} + \dot{L}_2 \mathbf{e}^{(2)} + \dot{L}_3 \mathbf{e}^{(3)} + L_1 \left[\boldsymbol{\omega} \times \mathbf{e}^{(1)} \right] + L_2 \left[\boldsymbol{\omega} \times \mathbf{e}^{(2)} \right] + L_3 \left[\boldsymbol{\omega} \times \mathbf{e}^{(3)} \right]. \end{aligned} \tag{119}$$

Using

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}^{(1)} + \omega_2 \mathbf{e}^{(2)} + \omega_3 \mathbf{e}^{(3)} \tag{120}$$

and $\left[\mathbf{e}^{(1)} \times \mathbf{e}^{(2)} \right] = \mathbf{e}^{(3)}$, etc., one obtains

$$\begin{aligned} \left[\boldsymbol{\omega} \times \mathbf{e}^{(1)} \right] &= \mathbf{e}^{(2)} \omega_3 - \mathbf{e}^{(3)} \omega_2 \\ \left[\boldsymbol{\omega} \times \mathbf{e}^{(2)} \right] &= \mathbf{e}^{(3)} \omega_1 - \mathbf{e}^{(1)} \omega_3 \\ \left[\boldsymbol{\omega} \times \mathbf{e}^{(3)} \right] &= \mathbf{e}^{(1)} \omega_2 - \mathbf{e}^{(2)} \omega_1. \end{aligned} \tag{121}$$

Substituting this into the equation for $\dot{\mathbf{L}}$ and using $L_\alpha = I_\alpha \omega_\alpha$, one obtains

$$\begin{aligned} \dot{\mathbf{L}} &= \left\{ \dot{L}_1 - L_2 \omega_3 + L_3 \omega_2 \right\} \mathbf{e}^{(1)} + \left\{ \dot{L}_2 - L_3 \omega_1 + L_1 \omega_3 \right\} \mathbf{e}^{(2)} + \left\{ \dot{L}_3 - L_1 \omega_2 + L_2 \omega_1 \right\} \mathbf{e}^{(3)} \\ &= \left\{ I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 \right\} \mathbf{e}^{(1)} + \left\{ I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_3 \omega_1 \right\} \mathbf{e}^{(2)} + \left\{ I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 \right\} \mathbf{e}^{(3)}. \end{aligned} \tag{122}$$

Finally, projecting the torque in Eq. (67) onto $\mathbf{e}^{(\beta)}$, one arrives at the set of famous Euler equations

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + K_1 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + K_2 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + K_3. \end{aligned} \tag{123}$$

Alternatively, these equations can be written with respect to the components of $\mathbf{L}$:

$$\begin{aligned} \dot{L}_1 &= -\left(\frac{1}{I_2} - \frac{1}{I_3}\right) L_2 L_3 + K_1 \\ \dot{L}_2 &= -\left(\frac{1}{I_3} - \frac{1}{I_1}\right) L_3 L_1 + K_2 \\ \dot{L}_3 &= -\left(\frac{1}{I_1} - \frac{1}{I_2}\right) L_1 L_2 + K_3. \end{aligned} \tag{124}$$

One can easily check that these equations conserve $\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2$ in the absence of torques. Note that even for a free body Euler equations are nonlinear.

As, in general, the torque depends on the Euler angles θ, ϕ, and ψ, to make the system of Euler equations closed and ready for solution one has to substitute ω_α by their expressions in terms of θ, ϕ, and ψ, Eq. (17). Of course, this immediately kills the beauty of the Euler equations and makes them similar to the Lagrange equations, Eq. (100). Both sets of equations are in general pretty cumbersome so that even for composing these equations using computer algebra is advisable. It should be possible to prove their equivalence that is more involved than the same for translational motion.

For a free body, $K_\alpha = 0$, the advantage of the Euler equations, with respect to the Lagrangian formalism, is that one can at first solve the equations of motion for ω_α and then substitute the solution into Eq. (17), thus obtaining a set of equations for the Euler angles that can be solved as the second step. This is how analytical solutions for a free asymmetric $I_1 \neq I_2 \neq I_3$ top are obtained in a standard way. However, for a torque-free body one can use Eq. (85) that gives a one-step solution of the problem.

A more important advantage of the Euler equations is that they can be readily written with respect of a non-holonomic contact point O' such as in the case of a sphere or disc rolling on a surface without slipping. Within this approach, one at first solves a pure rotational problem and at the second stage finds the motion of the center of mass, see example with the wheel on a plane below.

Let us consider now a free symmetric top, $I_1 = I_2$. The third line of Eq. (123) yields $\dot{\omega}_3 = 0$ thus $\omega_3 = const$. This makes the other two Euler equations linear:

$$\dot{\omega}_1 = \Omega\omega_2, \qquad \dot{\omega}_2 = -\Omega\omega_1, \qquad \Omega \equiv \left(1 - \frac{I_3}{I_1}\right)\omega_3. \tag{125}$$

The solution of these equations is

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right\} = \omega_\perp \left\{ \begin{array}{c} \sin(\Omega t + \varphi_0) \\ \cos(\Omega t + \varphi_0) \end{array} \right\}, \qquad \omega_\perp \equiv \sqrt{\omega_1^2 + \omega_2^2} = const, \tag{126}$$

that is, the vector $\boldsymbol{\omega}$ is precessing around $\mathbf{e}^{(3)}$ with frequency Ω. Comparison with Eq. (75) shows $\Omega = \dot{\psi}$, that is, the effect must be related to the rotation of the body around its symmetry axis $\mathbf{e}^{(3)}$. Note that in the derivation of Eq. (123) vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are considered as embedded in the body and rotating with it. Thus the rotation $\dot{\psi}$ automatically causes the vector $\boldsymbol{\omega}$ to precess in the body frame. This precession has little in common with precession of $\boldsymbol{\omega}$ with respect to the laboratory frame, Eq. (81), although some books say that these are two ways of viewing the same.

Precession described by Eq. (126) can be removed if one uses the axial symmetry of the body and employs sliding vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ instead of embedded ones, as was already done above, see the comment below

Eq. (60). With such a choice of $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ the derivation of the Euler equations has to be modified as rotation of the body around the symmetry axis does not result in the change of $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$. Thus in Eq. (119) one has to replace

$$\boldsymbol{\omega} \Rightarrow \boldsymbol{\omega}' \equiv \boldsymbol{\omega} - \dot{\psi}\mathbf{e}^{(3)}. \tag{127}$$

Then in Eq. (121) one replaces

$$\omega_3 \Rightarrow \omega_3' \equiv \omega_3 - \dot{\psi} = \dot{\phi}\cos\theta, \tag{128}$$

see Eq. (17). Thus instead of Eq. (122) one obtains

$$\begin{aligned}
\dot{\mathbf{L}} &= \left\{\dot{L}_1 - L_2\omega_3 + L_2\dot{\psi} + L_3\omega_2\right\}\mathbf{e}^{(1)} + \left\{\dot{L}_2 - L_3\omega_1 + L_1\omega_3 - L_1\dot{\psi}\right\}\mathbf{e}^{(2)} + \left\{\dot{L}_3 - L_1\omega_2 + L_2\omega_1\right\}\mathbf{e}^{(3)} \\
&= \left\{I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 + I_2\omega_2\dot{\psi}\right\}\mathbf{e}^{(1)} + \left\{I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 - I_1\omega_1\dot{\psi}\right\}\mathbf{e}^{(2)} \\
&\qquad + \left\{I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2\right\}\mathbf{e}^{(3)}.
\end{aligned} \tag{129}$$

With $I_1 = I_2$ this results in a special form of the Euler equations

$$\begin{aligned}
I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 - I_2\omega_2\dot{\psi} + K_1 \\
I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + I_1\omega_1\dot{\psi} + K_2 \\
I_3\dot{\omega}_3 &= K_3.
\end{aligned} \tag{130}$$

For a free symmetric top the third equation gives $\omega_3 = \text{const}$, thus the first two equations become

$$\begin{aligned}
\dot{\omega}_1 &= \left(\Omega - \dot{\psi}\right)\omega_2 = 0 \\
\dot{\omega}_2 &= -\left(\Omega - \dot{\psi}\right)\omega_1 = 0
\end{aligned} \tag{131}$$

with Ω given by Eq. (125). One can see that with the choice of the sliding internal frame for a symmetric top all components of $\boldsymbol{\omega}$ are constants and there is no precession of $\boldsymbol{\omega}$ in the body frame, as we have seen in Sec. 2.2.3.

One can make one more step and eliminate $\dot{\psi}$ in Eq. (130) using $\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$ that yields equations

$$\begin{aligned}
I_1\dot{\omega}_1 &= -I_3\omega_3\omega_2 + I_2\omega_2\dot{\phi}\cos\theta + K_1 \\
I_2\dot{\omega}_2 &= I_3\omega_3\omega_1 - I_1\omega_1\dot{\phi}\cos\theta + K_2 \\
I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + K_3
\end{aligned} \tag{132}$$

that are more convenient. In particular, if the components of the torque are independent of ψ, one can use ω_3 as a dynamic variable together with θ and ϕ. For $I_1 = I_2$ Eq. (132) simplifies to

$$\begin{aligned}
I_1\dot{\omega}_1 &= \left(-I_3\omega_3 + I_1\dot{\phi}\cos\theta\right)\omega_2 + K_1 \\
I_1\dot{\omega}_2 &= \left(I_3\omega_3 - I_1\dot{\phi}\cos\theta\right)\omega_1 + K_2 \\
I_3\dot{\omega}_3 &= K_3.
\end{aligned} \tag{133}$$

Eqs. (133) are less elegant than the standard Euler equations since they do not become closed equations of motion for $\boldsymbol{\omega}$ in the absence of torques but couple to ψ. On the other hand, in the presence of torques substituting Eq. (17) into the Euler equation becomes mandatory and the form of Euler equation such as Eqs. (133) can be of advantage. For instance, for a heavy symmetric top considered in Sec. 3.1.3 one has $K_3 = 0$ and the third equation yields $L_3 = I_3\omega_3 = \text{const}$. Then the first and second equations define the time dependence of θ and ϕ. With $\mathbf{e}^{(1)}$ always horizontal one has $K_1 = -Mga\sin\theta$ and $K_2 = 0$. Because of the symmetry around the z axis L_z given by Eq. (91) is conserved. This follows from Eq. (67) and Eq. (70) that yields $K_z = 0$. Using $L_z = \text{const}$ one can obtain $\dot{\phi}$ given by the first line of Eq. (104) and plug this result into the first equation, discarding the second equation. With the use of Eq. (61) the first equation becomes $I_1\ddot{\theta} = \ldots$ that concides with Eq. (106). Eq. (132) proves to be powerful in the case of a wheel rolling on a plane that will be considered below as an example.

3.2.2 Free asymmetric top

Insights in rather complicated motion of a free asymmetric top can be gained already from geometrical considerations based on conservation of the angular momentum $\mathbf{L}^2 = L^2 = \text{const}$ and energy $T_{\text{rot}} = E = \text{const}$. From Eqs. (64) and (65) one obtains

$$\begin{aligned} \frac{L_1^2}{L^2} + \frac{L_2^2}{L^2} + \frac{L_3^2}{L^2} &= 1 \\ \frac{L_1^2}{2EI_1} + \frac{L_2^2}{2EI_2} + \frac{L_3^2}{2EI_3} &= 1. \end{aligned} \tag{134}$$

The first of these equations represents a sphere of radius L and the second equation represents an ellipsoid with half-axes $\sqrt{2EI_1}$, $\sqrt{2EI_2}$, and $\sqrt{2EI_3}$. Vector $\mathbf{L}$ (or its terminus) can move along the lines in space that are defined by the intersection of the sphere and the ellipsoid. Let us assume, for a moment, $I_1 < I_2 < I_3$. Then for $L^2 < 2EI_1$ and $L^2 > 2EI_3$ there is no intersection, so that such values of L and E are unphysical. If L^2 slightly exceeds $2EI_1$ or is slightly below $2EI_3$, the intersection lines are small contours encircling $\mathbf{e}^{(1)}$ or $\mathbf{e}^{(3)}$. Thus the system prepared in such states will stay forever with directions of $\mathbf{L}$ near $\mathbf{e}^{(1)}$ or $\mathbf{e}^{(3)}$. One concludes that rotations around the principal axis with minimal and maximal moments of inertia are stable. This was also confirmed by careful experiments with a chalk wiper at the Graduate Center of CUNY. To obtain these small contours, one can subtract the two equations from each other and replace the appropriate component of $\mathbf{L}$ by L. For instance, if L^2 is slightly below $2EI_3$, one obtains

$$\begin{aligned} 0 &= \left(\frac{1}{L^2} - \frac{1}{2EI_1}\right) L_1^2 + \left(\frac{1}{L^2} - \frac{1}{2EI_2}\right) L_2^2 + \left(\frac{1}{L^2} - \frac{1}{2EI_3}\right) L_3^2 \\ &\cong \left(\frac{1}{2EI_3} - \frac{1}{2EI_1}\right) L_1^2 + \left(\frac{1}{2EI_3} - \frac{1}{2EI_2}\right) L_2^2 + \left(\frac{1}{L^2} - \frac{1}{2EI_3}\right) L^2, \end{aligned} \tag{135}$$

or

$$\left(\frac{I_3}{I_1} - 1\right) L_1^2 + \left(\frac{I_3}{I_2} - 1\right) L_2^2 = 2EI_3 - L^2. \tag{136}$$

This is an equation of a small (L_1, L_2) ellipse with half-axes

$$\sqrt{(2EI_3 - L^2)\,\frac{I_1}{I_3 - I_1}}, \qquad \sqrt{(2EI_3 - L^2)\,\frac{I_2}{I_3 - I_2}}, \tag{137}$$

whereas $L_3 \cong L$. If I_3 is the intermediate moment of inertia, say, $I_1 < I_3 < I_2$, Eq. (136) describes hyperboles since the coefficients in front of L_1^2 and L_3^2 have different signs. Thus with time the top strongly deviates from the initial direction of rotation and Eq. (136) loses its validity.

The dynamics of the above phenomena can be studied with the help of the Euler equations for $\boldsymbol{\omega}$ or $\mathbf{L}$. Let us take Eq. (123) with $\mathbf{K} = \mathbf{0}$ and consider rotations around the directions close to $\mathbf{e}^{(3)}$ so that ω_1 and ω_2 are small and $\omega_1\omega_2$ in the third equation can be neglected. Then $\omega_3 \cong \text{const}$ and instead of Eq. (125) one obtains

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3, \qquad \dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_1\omega_3. \tag{138}$$

Combining the two equations yields

$$\ddot{\omega}_1 + \Omega^2\omega_1 = 0, \qquad \Omega = \sqrt{\left(\frac{I_3}{I_1} - 1\right)\left(\frac{I_3}{I_2} - 1\right)}\omega_3 \tag{139}$$

and the same equation for ω_2. If I_3 is the maximal or minimal moment of inertia, Ω is real and both ω_1 and ω_2 perform oscillations with frequency Ω. In fact, $\boldsymbol{\omega}$ performs elliptic precession around $\mathbf{e}^{(3)}$, and $\mathbf{L}$ does the similar. Elliptic character of precession is clear from Eq. (136). If, however, I_3 is the intermediate moment of inertia, Ω is imaginary and both ω_1 and ω_2 exponentially grow with time from small initial values and soon become comparable with ω_3 so that the approximation breaks down.

In the general case the solution for ω_α or L_α in the body frame can be obtained in terms of Jacobian elliptic functions. The solution is periodic with a period T being an elliptic integral depending on E and L. After that the Euler angles θ and ψ can be found from Eq. (84). As the angle ϕ does not enter these equations, it has to be found by integration using, for instance, Eq. (17) that can be resolved for $\dot{\phi}$. The result of this very complicated calculation is that ϕ behaves periodically with a period T' that is incommensurate with T. Because of this, the free asymmetric top never returns to a previous orientation.

Eq. (85) provides a direct access to the time-dependent orientation of a free asymmetric top, bypassing the Euler equations. Consider, for instance, an orientation in the vicinity of $\theta = \psi = \pi/2$, i.e., $\mathbf{e}^{(1)} = \mathbf{e}_z$ and $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$ in the $x - y$ plane, dependent on ϕ. Since $\mathbf{L}\|\mathbf{e}_z$, this means that the top is rotating around the axis with the moment of inertia I_1. Deviations from this orientation can be described by

$$\theta = \pi/2 + \delta\theta, \qquad \psi = \pi/2 + \delta\psi \tag{140}$$

with small $\delta\theta$ and $\delta\psi$. Linearizing the equations for $\dot{\theta}$ and $\dot{\psi}$ yields

$$\begin{aligned} \frac{d}{dt}\delta\theta &= -\left(\frac{1}{I_1} - \frac{1}{I_2}\right) L\delta\psi \\ \frac{d}{dt}\delta\psi &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right) L\delta\theta. \end{aligned} \tag{141}$$

These equations describe a small precession of $\mathbf{e}^{(1)}$ around $\mathbf{e}_z$ with the frequency

$$\Omega = \sqrt{\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\left(\frac{1}{I_1} - \frac{1}{I_3}\right)} L = \sqrt{\left(1 - \frac{I_1}{I_2}\right)\left(1 - \frac{I_1}{I_3}\right)} \omega_1, \tag{142}$$

similar to Eq. (139). The precession rate $\dot{\phi}$ in this approximation is given by its unperturbed value

$$\dot{\phi} = \frac{L}{I_1} = \omega_1 \tag{143}$$

corresponding to $\psi = \pi/2$. One can see that, indeed, the top performs two motions with incommensurate frequencies Ω and ω_1. The angular deviations $\delta\theta$ and $\delta\psi$ define the small projection of $\mathbf{e}^{(1)}$ onto the $x - y$ plane that makes loops with frequency Ω. On the other hand, small projections of $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$ onto $\mathbf{e}_z$, also proportional to $\delta\theta$ and $\delta\psi$, depend additionally on ϕ and make a motion with combinational frequencies $\Omega \pm \omega_1$. The trajectory of $\delta\theta$ and $\delta\psi$ can be found from Eq. (88). Linearization yields

$$\begin{aligned} \frac{2E}{L^2} &\cong \frac{\left(1 - (\delta\theta)^2/2\right)^2 \left(1 - (\delta\psi)^2/2\right)^2}{I_1} + \frac{(\delta\psi)^2}{I_2} + \frac{(\delta\theta)^2}{I_3} \\ &\cong \frac{1}{I_1} + \left(\frac{1}{I_3} - \frac{1}{I_1}\right)(\delta\theta)^2 + \left(\frac{1}{I_2} - \frac{1}{I_1}\right)(\delta\psi)^2 \end{aligned} \tag{144}$$

or

$$\left(1 - \frac{I_1}{I_3}\right)(\delta\theta)^2 + \left(1 - \frac{I_1}{I_2}\right)(\delta\psi)^2 = 1 - \frac{2EI_1}{L^2}. \tag{145}$$

This describes an $(\delta\theta, \delta\psi)$ ellipse with half-axes

$$\sqrt{\left(1 - \frac{2EI_1}{L^2}\right)\frac{I_3}{I_3 - I_1}}, \qquad \sqrt{\left(1 - \frac{2EI_1}{L^2}\right)\frac{I_2}{I_2 - I_1}} \tag{146}$$

c.f. Eq. (137).

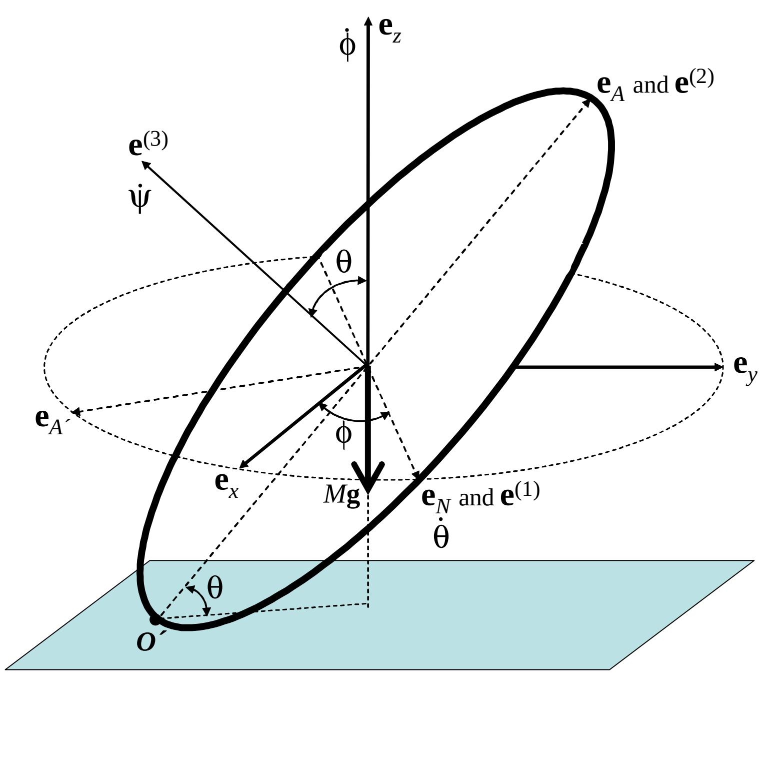
One can do a similar analysis around the orientation $\theta = \pi/2$ and $\psi = 0$, i.e., $\mathbf{e}^{(2)} = \mathbf{e}_z$ and $\mathbf{e}^{(3)}$, $\mathbf{e}^{(1)}$ in the $x-y$ plane. In this case the top rotates around $\mathbf{e}^{(2)}$ and for $I_1 < I_2 < I_3$ this rotation is unstable. As a result, Ω is imaginary and small initial deviations exponentially increase. Instead of Eq. (145) one obtains an equation with different signs at $(\delta\theta)^2$ and $(\delta\psi)^2$ that describe hyperboles.

It should be noted that the method becomes unconvenient for the orientation in the vicinity of $\theta = 0$ i.e., $\mathbf{e}^{(3)} = \mathbf{e}_z$. In this case ψ is not confined to the vicinity of a particular value and the equations cannot be linearized in ψ. This is the well-known singularity of the spherical coordinate system. The easiest way to avoid this formal difficulty and consider the case of rotation around the axis with the maximal moment of inertia is just to assume that the latter is I_1 rather than I_3 and use Eq. (140).

An important implication of the analysis in this section is that the Larmor description of a rapidly rotating top introduced in Sec. 2.2.5 is only possible if $\mathbf{L}$ is nearly parallel to the axis with the maximal or minimal moment of inertia. Only in these cases the motion is stable and the initial assumption of Eq. (94) is valid.

4 Advanced problems

4.1 Wheel on a plane



4.1.1 Equations of motion

Let us consider, as an illustration, a wheel in form of a disk or a ring of radius R and mass M with moments of inertia

$$I_1 = I_2 = I/2, \qquad I_3 = I \tag{147}$$

rolling without slipping on a plane, taking into account the gravity force and a constant applied force

$$\mathbf{F} = -Mg\mathbf{e}_z + F_x\mathbf{e}_x. \tag{148}$$

The applied force may be, for instance, due to the inclination of the plane, $F_x = Mg\sin\alpha$. The rolling constraint in our case

$$\mathbf{V} = R\left[\boldsymbol{\omega}\times\mathbf{e}^{(2)}\right] \tag{149}$$

is non-holonomic as it cannot be eliminated by integration. Here the Lagrange formalism becomes cumbersome and the Newtonian approach is of advantage.

We use the sliding vectors $\mathbf{e}^{(1)} = \mathbf{e}_N$ and $\mathbf{e}^{(2)} = \mathbf{e}_A$, as shown in the Figure, the particular form of Eq. (21) with $\psi = 0$

$$\begin{aligned} \mathbf{e}^{(1)} &= \cos\phi\mathbf{e}_x + \sin\phi\mathbf{e}_y \\ \mathbf{e}^{(2)} &= -\cos\theta\sin\phi\mathbf{e}_x + \cos\theta\cos\phi\mathbf{e}_y + \sin\theta\mathbf{e}_z \\ \mathbf{e}^{(3)} &= \sin\theta\sin\phi\mathbf{e}_x - \sin\theta\cos\phi\mathbf{e}_y + \cos\theta\mathbf{e}_z. \end{aligned} \tag{150}$$

The Euler equations with respect to the instantaneous contact point (CP) O' and using have the form similar to Eq. (132)

$$\begin{aligned} I_1'\dot{\omega}_1 &= -I_3'\omega_3\omega_2 + I_2\omega_2\dot{\phi}\cos\theta + K_1' \\ I_2\dot{\omega}_2 &= I_3'\omega_3\omega_1 - I_1'\omega_1\dot{\phi}\cos\theta + K_2' \\ I_3'\dot{\omega}_3 &= (I_1' - I_2)\omega_1\omega_2 + K_3'. \end{aligned} \tag{151}$$

Here

$$I_1' = I/2 + MR^2, \qquad I_2 = I/2, \qquad I_3' = I + MR^2 \tag{152}$$

are shifted principal moments of inertia with respect to the instantaneous rotation axis going through O'. Note that, although the wheel is rotationally symmetric, the shifted moments of inertia are all different, so that the problem is of the same grade of difficulty as that of a fully asymmetric body. Because of the rotational symmetry of the wheel, one can choose the vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ so that $\mathbf{e}^{(1)}$ is horizontal and $\mathbf{e}^{(2)}$ is directed from CP to CM, as shown in the Figure. Of course, $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are not embedded in the body but slide with respect to it. The torque $\mathbf{K}'$ is given by

$$\begin{aligned} \mathbf{K}' &= \left[R\mathbf{e}^{(2)}\times\mathbf{F}\right] = -MgR\left[\mathbf{e}^{(2)}\times\mathbf{e}_z\right] + F_xR\left[\mathbf{e}^{(2)}\times\mathbf{e}_x\right] \\ &= -MgR\cos\theta\mathbf{e}^{(1)} + F_xR\left[\mathbf{e}^{(2)}\times\left(\cos\phi\mathbf{e}^{(1)} + \sin\phi\mathbf{e}_{A'}\right)\right] \\ &= -MgR\cos\theta\mathbf{e}^{(1)} - F_xR\cos\phi\mathbf{e}^{(3)} + F_xR\sin\theta\sin\phi\mathbf{e}^{(1)}, \end{aligned} \tag{153}$$

where

$$\mathbf{e}_{A'} = \mathbf{e}^{(3)}\sin\theta - \mathbf{e}^{(2)}\cos\theta \tag{154}$$

is another antinode vector, see Figure. In components this result reads

$$\begin{aligned} K_1' &= -MgR\cos\theta + F_xR\sin\theta\sin\phi \\ K_2' &= 0 \\ K_3' &= -F_xR\cos\phi. \end{aligned} \tag{155}$$

The applied force tends to accelerate rolling of the wheel if $\phi = 0$ and to cause its falling for $\phi = \pi/2$, together with the gravity force.

Since the torque depends on the Euler angles, one has express $\boldsymbol{\omega}$ in therms of the Euler angles, too. This is done with the help of Eq. (61) that employs $\psi = 0$. However, it turns out that one can take $L_3 = I_3'\omega_3$ as one of dynamical variables, thus reducing the order of derivatives in equations. To this end, we eliminate $\dot{\psi}$ in Eq. (151) using $\dot{\psi} = \omega_3 - \dot{\phi}\cos\theta$. This yields a system of nonlinear differential equations

$$\begin{aligned} I_1'\ddot{\theta} &= \left(-L_3 + I_2\dot{\phi}\cos\theta\right)\dot{\phi}\sin\theta - MgR\cos\theta + F_xR\sin\theta\sin\phi \\ I_2\frac{d}{dt}\left(\dot{\phi}\sin\theta\right) &= \left(L_3 - I_1'\dot{\phi}\cos\theta\right)\dot{\theta} \\ \dot{L}_3 &= MR^2\dot{\theta}\dot{\phi}\sin\theta - F_xR\cos\phi. \end{aligned} \tag{156}$$

This system of equations is complicated and in general it should be solved numerically.

After the solution of this system of equations is found, one can obtain the velocity of the center of mass from Eq. (149):

$$\mathbf{V} = \left[\boldsymbol{\omega} \times R\mathbf{e}^{(2)}\right] = R\left[\left(\omega_1 \mathbf{e}^{(1)} + \omega_3 \mathbf{e}^{(3)}\right) \times \mathbf{e}^{(2)}\right] = R\omega_1 \mathbf{e}^{(3)} - R\omega_3 \mathbf{e}^{(1)}. \tag{157}$$

Using Eq. (150), one finally obtains

$$\mathbf{V} = R\left(\omega_1 \sin\theta \sin\phi - \omega_3 \cos\phi\right) \mathbf{e}_x + R\left(-\omega_1 \sin\theta \cos\phi - \omega_3 \sin\phi\right) \mathbf{e}_y + R\omega_1 \cos\theta \mathbf{e}_z. \tag{158}$$

With $\omega_1 = \dot{\theta}$, the first two components of this yield differential equations

$$\begin{aligned} \dot{X} &= R\left(\dot{\theta} \sin\theta \sin\phi - \frac{L_3}{I_3'} \cos\phi\right) \\ \dot{Y} &= -R\left(\dot{\theta} \sin\theta \cos\phi + \frac{L_3}{I_3'} \sin\phi\right), \end{aligned} \tag{159}$$

whereas the third differential equation can be immediately integrated:

$$Z = R \sin\theta. \tag{160}$$

It is interesting to note that the z-components of the non-holonomic constraint, Eq. (149), is holonomic since it can be integrated!

The position of the contact point $\boldsymbol{\xi}$ is related to that of the CM

$$\mathbf{R} = X\mathbf{e}_x + Y\mathbf{e}_y + Z\mathbf{e}_z \tag{161}$$

by the formula

$$\boldsymbol{\xi} = \mathbf{R} - R\mathbf{e}^{(2)} \tag{162}$$

(do not confuse $\mathbf{R}$ with the wheel radius R). The components of this formula are

$$\begin{aligned} \xi_x &= X + R\cos\theta \sin\phi \\ \xi_y &= Y - R\cos\theta \cos\phi. \end{aligned} \tag{163}$$

In the case of the forces that depend on the wheel's horizontal position X,Y the problem cannot be solved in two steps. In this case one has to solve Eq. (156) with appropriately modified torques together with Eq. (159).

4.1.2 Analytical considerations and results

In the case $F_x = 0$ there is a rotational symmetry around the z axis and one has $L_z = \text{const}$. Using Eq. (91) in the form

$$L_z = I_2 \dot{\phi} \sin^2\theta + L_3 \cos\theta, \tag{164}$$

one can express $\dot{\phi}$ through L_3,

$$\dot{\phi} = \frac{L_z - L_3 \cos\theta}{I_2 \sin^2\theta}, \tag{165}$$

c.f. Eq. (104), substitute it into the first and third lines of Eq. (156), and discard the second line Eq. (156). This yields the system of equations

$$\begin{aligned} I_1' \ddot{\theta} &= \frac{\left(L_z - L_3 \cos\theta\right)\left(L_z \cos\theta - L_3\right)}{I_2 \sin^3\theta} - MgR\cos\theta \\ \dot{L}_3 &= MR^2 \dot{\theta} \frac{L_z - L_3 \cos\theta}{I_2 \sin\theta}. \end{aligned} \tag{166}$$

One can see that the first equation can be represented in the form of Eq. (107) with $U_{\text{eff}}(\theta)$ given by Eq. (109) with a modified form of the gravity energy. The energy of the θ motion (nutation)

$$E_\theta = \frac{1}{2} I_1' \dot{\theta}^2 + U_{\text{eff}}(\theta) = \frac{1}{2} I_1' \dot{\theta}^2 + \frac{(L_z - L_3 \cos\theta)^2}{2I_1' \sin^2\theta} + MgR\sin\theta \tag{167}$$

is not conserved since L_3 in $U_{\text{eff}}(\theta)$ depends on time according to Eq. (166). On the other hand, the total energy $E = T_{\text{rot}} + U$ is conserved,

$$E = \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2} I_2' \dot{\phi}^2 \sin^2\theta + \frac{L_3^2}{2I_3'} + MgR\sin\theta = E_\theta + \frac{L_3^2}{2I_3'} = \text{const.} \tag{168}$$

The energy of the 3-rotation $L_3^2/(2I_3')$ is converted into the energy of nutation E_θ and vice versa. One can eliminate L_3 using Eq. (168) and obtain a closed equation of motion for θ of the type $I_1' \ddot{\theta} = \ldots$ However, the rhs of this equation contains $\dot{\theta}^2$ that makes the torque acting on θ non-conservative, so that one cannot introduce an effective energy for this torque and simplify the problem.

Eq. (166) allows one to find the stability region for a wheel rolling straight, $L_z = 0$, and upright, $\theta = \pi/2$. Expanding the equation for θ with the help

$$\theta = \pi/2 + \delta\theta, \qquad \sin\theta \cong 1, \qquad \cos\theta \cong -\delta\theta \tag{169}$$

one obtains

$$I_1' \frac{d^2}{dt^2} \delta\theta \cong \left(MgR - \frac{L_3^2}{I_2} \right) \delta\theta \tag{170}$$

Thus stability of wheel's upright rolling requires

$$L_3^2 > I_2 MgR, \tag{171}$$

that resembles Eq. (112). The second line of Eq. (166) becomes

$$\dot{L}_3 \cong \frac{MR^2}{2I_2} L_3 \frac{d}{dt} (\delta\theta)^2 . \tag{172}$$

That is, if the wheel loses stability and begins to fall on a side, L_3 increases. This quadratic effect cannot change the stability criterium, Eq. (171), obtained from linear analysis.

The wheel also can roll stationary ($\theta = \text{const}$, $L_3 = \text{const}$) with $L_z \neq 0$ and $\theta \neq \pi/2$. In the absence of gravity the first line of Eq. (166) yields $\cos\theta = L_z/L_3$. According to Eq. (164) this implies $\dot{\phi} = 0$, i.e., the wheel is still rolling straight. Adding gravity causes $\dot{\phi} \neq 0$ and the wheel making circles.

If the wheel is rapidly rotationg around its symmetry axis, the problem simplifies and can be solved by a method similar to that of Sec. 2.2.5. Again, L_3 dominates in the angular momentum and one can use the approximate Eq. (94) that implies

$$\mathbf{e}^{(3)} = \mathbf{L}/L \tag{173}$$

(also $\mathbf{e}^{(3)} = \boldsymbol{\omega}/\omega$). Using

$$\mathbf{e}^{(1)} = \frac{\left[\mathbf{e}_z \times \mathbf{e}^{(3)}\right]}{\sqrt{\left[\mathbf{e}_z \times \mathbf{e}^{(3)}\right]^2}} = \frac{[\mathbf{e}_z \times \mathbf{L}]}{\sqrt{[\mathbf{e}_z \times \mathbf{L}]^2}} \tag{174}$$

and

$$\mathbf{e}^{(2)} = \left[\mathbf{e}^{(3)} \times \mathbf{e}^{(1)}\right] = \frac{1}{L} \left[\mathbf{L} \times \mathbf{e}^{(1)}\right] = \frac{1}{L} \frac{[\mathbf{L} \times [\mathbf{e}_z \times \mathbf{L}]]}{\sqrt{[\mathbf{e}_z \times \mathbf{L}]^2}} = \frac{1}{L} \frac{\mathbf{e}_z L^2 - \mathbf{L}\,(\mathbf{e}_z \cdot \mathbf{L})}{\sqrt{[\mathbf{e}_z \times \mathbf{L}]^2}}. \tag{175}$$

Now Eq. (67) with the torque given by Eq. (153) becomes

$$\dot{\mathbf{L}} = \mathbf{K}' = \left[R\mathbf{e}^{(2)} \times \mathbf{F}\right] = \frac{R}{L} \frac{L^2\,[\mathbf{e}_z \times \mathbf{F}] - (\mathbf{e}_z \cdot \mathbf{L})\,[\mathbf{L} \times \mathbf{F}]}{\sqrt{[\mathbf{e}_z \times \mathbf{L}]^2}} \tag{176}$$

or

$$\dot{\mathbf{L}} = \frac{(\mathbf{e}_z \cdot \mathbf{L}) [\mathbf{L} \times \boldsymbol{\Omega}] - L^2 [\mathbf{e}_z \times \boldsymbol{\Omega}]}{\sqrt{[\mathbf{e}_z \times \mathbf{L}]^2}}, \qquad \boldsymbol{\Omega} \equiv -\frac{R\mathbf{F}}{L} \tag{177}$$

that is a kind of Larmor equation (95) for the wheel.

If there is only the gravity force, $\mathbf{F} = -Mg\mathbf{e}_z$, our equation simplifies to

$$\dot{\mathbf{L}} = \frac{(\mathbf{e}_z \cdot \mathbf{L})}{\sqrt{[\mathbf{e}_z \times \mathbf{L}]^2}} [\mathbf{L} \times \boldsymbol{\Omega}], \qquad \boldsymbol{\Omega} = \frac{MgR}{L}\mathbf{e}_z, \tag{178}$$

that is, the angular momentum $\mathbf{L}$ of the wheel is precessing in different direction with respect to the heavy symmetric top and also with $L_z =$ const and $L^2 =$ const. That is, $\theta =$ const and the precession rate is

$$\dot{\phi} = -\Omega \cot\theta = -\frac{MgR}{L} \cot\theta. \tag{179}$$

This rate obviously turns to zero for the upright wheel, $\theta = \pi/2$, since the gravity torque disappears in this position. The upright wheel rolls straight. On the other hand, the precession rate becomes very large for nearly lying wheel, $\theta \approx 0, \pi$. Everybody have seen that wheels falling on a side begin to rotate very fast just before they hit the ground. Of course, the latter is caused by slipping that takes place in a more realistic model.

In the general case $L^2 \neq$ const and for the change of L from Eq. (177) one obtains the equation

$$\frac{d}{dt}L^2 = 2L\dot{L} = 2\mathbf{L} \cdot \dot{\mathbf{L}} = \frac{2L^2}{\sqrt{[\mathbf{e}_z \times \mathbf{L}]^2}} (\mathbf{L} \cdot [\boldsymbol{\Omega} \times \mathbf{e}_z]) \tag{180}$$

or

$$\dot{L} = \frac{L (\mathbf{L} \cdot [\boldsymbol{\Omega} \times \mathbf{e}_z])}{\sqrt{[\mathbf{e}_z \times \mathbf{L}]^2}} = L\frac{(\mathbf{n}_L \cdot [\boldsymbol{\Omega} \times \mathbf{e}_z])}{\sqrt{[\mathbf{e}_z \times \mathbf{n}_L]^2}} = L\frac{(\mathbf{e}_z \cdot [\mathbf{n}_L \times \boldsymbol{\Omega}])}{\sqrt{[\mathbf{e}_z \times \mathbf{n}_L]^2}}, \tag{181}$$

where we introduced $\mathbf{n}_L$ as is the unit vector defining the direction of $\mathbf{L}$,

$$\mathbf{L} \equiv L\mathbf{n}_L. \tag{182}$$

Using the definition of $\boldsymbol{\Omega}$, one can rewrite Eq. (181) as

$$\dot{L} = -\frac{(\mathbf{n}_L \cdot [R\mathbf{F} \times \mathbf{e}_z])}{\sqrt{[\mathbf{e}_z \times \mathbf{n}_L]^2}}. \tag{183}$$

This means that for a rapidly rotating wheel only the horizontal force can change L.

Let us now obtain the equation of motion for $\mathbf{n}_L$. In terms of $\mathbf{n}_L$ Eq. (177) takes the form

$$\dot{\mathbf{L}} = L\frac{(\mathbf{e}_z \cdot \mathbf{n}_L) [\mathbf{n}_L \times \boldsymbol{\Omega}] - [\mathbf{e}_z \times \boldsymbol{\Omega}]}{\sqrt{[\mathbf{e}_z \times \mathbf{n}_L]^2}}. \tag{184}$$

Differentiating Eq. (182) one obtains

$$\dot{\mathbf{L}} = \dot{L}\mathbf{n}_L + L\dot{\mathbf{n}}_L \tag{185}$$

and

$$\dot{\mathbf{n}}_L = \frac{\dot{\mathbf{L}}}{L} - \frac{\dot{L}}{L}\mathbf{n}_L = \frac{(\mathbf{e}_z \cdot \mathbf{n}_L) [\mathbf{n}_L \times \boldsymbol{\Omega}] - [\mathbf{e}_z \times \boldsymbol{\Omega}]}{\sqrt{[\mathbf{e}_z \times \mathbf{n}_L]^2}} - \frac{(\mathbf{n}_L \cdot [\boldsymbol{\Omega} \times \mathbf{e}_z])}{\sqrt{[\mathbf{e}_z \times \mathbf{n}_L]^2}}\mathbf{n}_L$$

$$= \frac{(\mathbf{e}_z\cdot\mathbf{n}_L)\,[\mathbf{n}_L\times\boldsymbol{\Omega}] - [\mathbf{e}_z\times\boldsymbol{\Omega}] + (\mathbf{e}_z\cdot[\boldsymbol{\Omega}\times\mathbf{n}_L])\,\mathbf{n}_L}{\sqrt{[\mathbf{e}_z\times\mathbf{n}_L]^2}}$$

$$= \frac{-[\mathbf{e}_z\times\boldsymbol{\Omega}] - [\mathbf{e}_z\times[\mathbf{n}_L\times[\mathbf{n}_L\times\boldsymbol{\Omega}]]]}{\sqrt{[\mathbf{e}_z\times\mathbf{n}_L]^2}}$$

$$= \frac{\boldsymbol{\Omega}+[\mathbf{n}_L\times[\mathbf{n}_L\times\boldsymbol{\Omega}]]}{\sqrt{[\mathbf{e}_z\times\mathbf{n}_L]^2}}\times\mathbf{e}_z = \frac{\mathbf{n}_L\,(\mathbf{n}_L\cdot\boldsymbol{\Omega})}{\sqrt{[\mathbf{e}_z\times\mathbf{n}_L]^2}}\times\mathbf{e}_z \tag{186}$$

or

$$\dot{\mathbf{n}}_L = \frac{(\mathbf{n}_L\cdot\boldsymbol{\Omega})}{\sqrt{[\mathbf{e}_z\times\mathbf{n}_L]^2}}\,[\mathbf{n}_L\times\mathbf{e}_z] \tag{187}$$

c.f. Eq. (177). The result of this long calculation is rewarding. One can see that vector $\mathbf{n}_L$ is precessing around $\mathbf{e}_z$ so that $(\mathbf{e}_z\cdot\mathbf{n}_L) = \cos\theta = \text{const}$. This does not imply $L_z = \text{const}$, however, since the length of the precessing vector $\mathbf{L}$ changes according to Eq. (181). The rate of this precession is just $\dot{\phi}$ and it is not constant because $(\mathbf{n}_L\cdot\boldsymbol{\Omega})$ depends on the direction of $\mathbf{n}_L$, so that even the direction of precession can change. There are two regimes: (i) Weak horizontal force, $\mathbf{n}_L$ precessing in one direction; (ii) Strong horizontal force, $\mathbf{n}_L$ is precessing in different directions keeping some average orientation. As

$$(\mathbf{n}_L\cdot\boldsymbol{\Omega}) \propto (\mathbf{n}_L\cdot\mathbf{F}) = -Mg\cos\theta + F_x\sin\theta\cos\phi, \tag{188}$$

the condition that $(\mathbf{n}_L\cdot\boldsymbol{\Omega})$ never changes sign [the weak-force regime (i)] is

$$F_x < Mg\cot\theta, \tag{189}$$

while the strong-force regime (i) is the opposite. For the upright wheel $\theta = \pi/2$ and regime (ii) is realized for whatever small force F_x.

If there is only gravity force, then $\boldsymbol{\Omega}$ is vertical and $\mathbf{n}_L$ is precessing at constant rate, as we have seen above. According to Eqs. (181) or (183), $L = \text{const}$. In regime (i) $\dot{L}$ changes sign and $L = \text{const}$ on average. In regime (ii), $\mathbf{n}_L$ is precessing in different directions so that on average $(\mathbf{n}_L\cdot\boldsymbol{\Omega}) = 0$. This means that on average $\mathbf{F}$ is in the plane of the wheel and the wheel rolls on average in the direction of the horizontal force. The orientation $(\mathbf{n}_L\cdot\boldsymbol{\Omega}) = 0$ maximizes $\dot{L}$ in Eq. (181), so that the wheel accelerates in the direction of the applied force. Eq. (183) shows that L grows linearly with time in this case.

To find the motion of the CM of the disk, we use Eq. (149). Using

$$\boldsymbol{\omega} = \omega\mathbf{e}^{(3)} = \mathbf{L}/I_3 \tag{190}$$

for the rapidly rotating top and Eq. (175), one obtains

$$\mathbf{V} = \left[\boldsymbol{\omega}\times R\mathbf{e}^{(2)}\right] = \frac{RL}{I_3}\frac{[\mathbf{L}\times\mathbf{e}_z]}{\sqrt{[\mathbf{L}\times\mathbf{e}_z]^2}} = \frac{RL}{I_3}\frac{[\mathbf{n}_L\times\mathbf{e}_z]}{\sqrt{[\mathbf{n}_L\times\mathbf{e}_z]^2}}, \tag{191}$$

where the dynamics of $\mathbf{n}_L$ are governed by Eq. (187). In general, this equation should be integrated numerically to obtain the horizontal component $\mathbf{R}_{XY}$ of the wheel's position vector $\mathbf{R}$ of Eq. (161). In the absence of the horizontal force using Eq. (187) with $(\mathbf{n}_L\cdot\boldsymbol{\Omega}) = \Omega\cos\theta$ Eq. (191) simplifies to

$$\mathbf{V} = \frac{RL}{I_3\Omega}\frac{\dot{\mathbf{n}}_L}{\cos\theta} = \frac{L^2}{MgI_3}\frac{\dot{\mathbf{n}}_L}{\cos\theta} \tag{192}$$

and can be immediately integrated resulting in

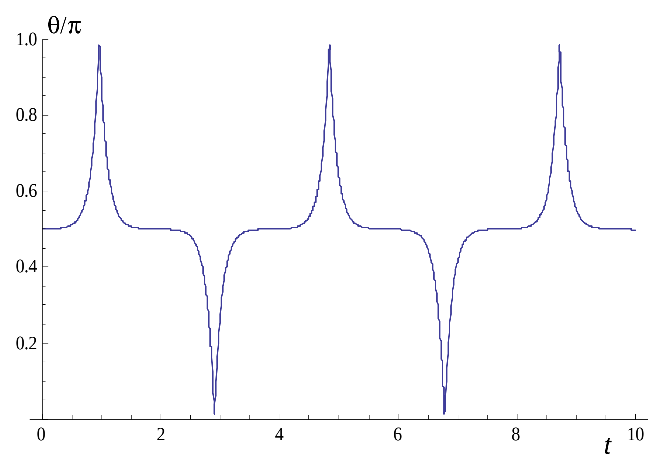
$$\mathbf{R}_{XY} = \frac{L^2}{MgI_3}\frac{\mathbf{n}_L - \mathbf{e}_z\,(\mathbf{e}_z\cdot\mathbf{n}_L)}{\cos\theta} = \frac{L^2}{MgI_3}\tan\theta\,\frac{\mathbf{n}_L - \mathbf{e}_z\,(\mathbf{e}_z\cdot\mathbf{n}_L)}{|\mathbf{n}_L - \mathbf{e}_z\,(\mathbf{e}_z\cdot\mathbf{n}_L)|}. \tag{193}$$

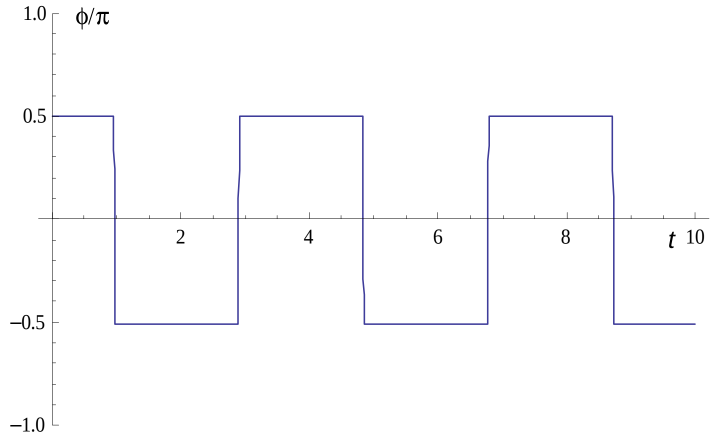
The factor in front of the rotating horizontal unit vector in this expression is the radius of the circular orbit of the wheel's CM. In regime (i) with nonzero horizontal force the wheel is making loops without accelerating on average and drifting perpendicular to the horizontal force. The realization of the two possible directions in this regime depends on initial conditions. In regime (ii) $[\mathbf{e}_z \times \mathbf{n}_L]$ in Eq. (191) has a nonzero average value and the wheel accelerates in the direction of the applied horizontal force.

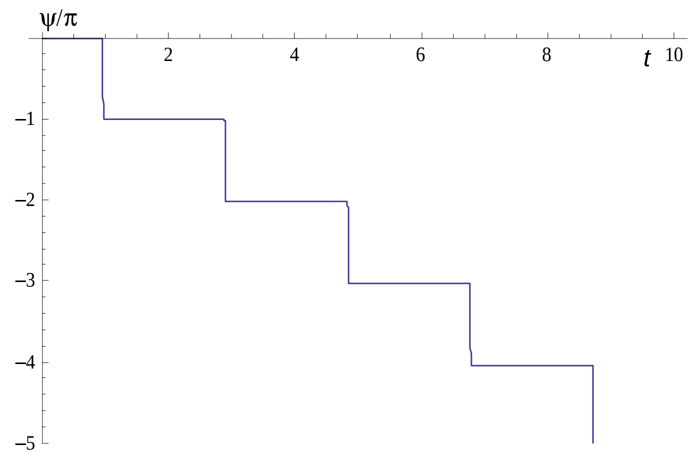
4.1.3 Numerical results

Numerical solution of the wheel-on-a-plane problem implemented in Wolfram Mathematica can be downloaded separately. The parameters of the wheel are set to $M = R = I = 1$. The results show that a rolling wheel never falls on a side in spite of the gravity torque. Rolling with the rotation around the symmetry axis is pretty stable. Applied force F_x tends to accelerate the wheel's motion in the direction of the force.

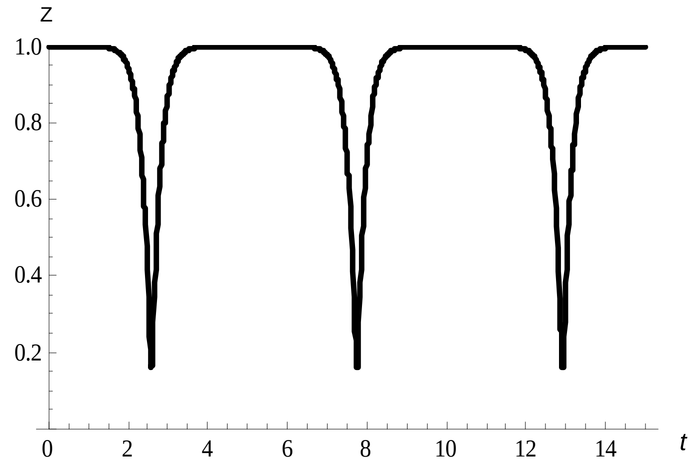
If the wheel's initial rotation is very slow, it nearly falls flat but, as θ approaches 0 or π, the rotation dramatically increases so that both $\dot{\phi}$ and $\dot{\psi}$ become large and the sign of $\dot{\theta}$ gets reversed. During the short nearly-fall time interval both the CM and the contact point assume large velocities and the contact point makes a bow around the CM. Then the wheel stands up again until the next fall on one of the sides. One of the numerical solutions in the nearly falling regime is presented below The forces are $Mg = 100$ and $F_x = 0$. The initial conditions were $X(0) = Y(0) = 0$, $\theta(0) = \pi/2$, $\phi(0) = \pi/2$ and $\dot{\theta}(0) = 0.01$, $\dot{\phi}(0) = 0$, $\dot{\psi}(0) = -0.001$. That is, in the initial state the wheel is upright, its plane is parallel to the y-axis and it begins rolling very slowly in the positive y direction because $\dot{\psi}(0) < 0$. However, a small push in the positive x direction, $\dot{\theta}(0) = 0.01$, together with the gravity torque cause the wheel to nearly fall flat. The CM begins to move to the right and almost reaches the surface. But because of the small initial rotation the derivatives $\dot{\phi}$ and $\dot{\psi}$ strongly increase and the wheel makes a fast rotation with displacement stands up again. The first figire below shows the dependence $\theta(t)$ with nearly falling on both sides and recovering. The second figure shows that the wheel is rapidly precessing in alternating directions when the wheel nearly hits the ground. The third figure shows that at these moments the wheel is rapidly rotating around its symmetry axis, too.

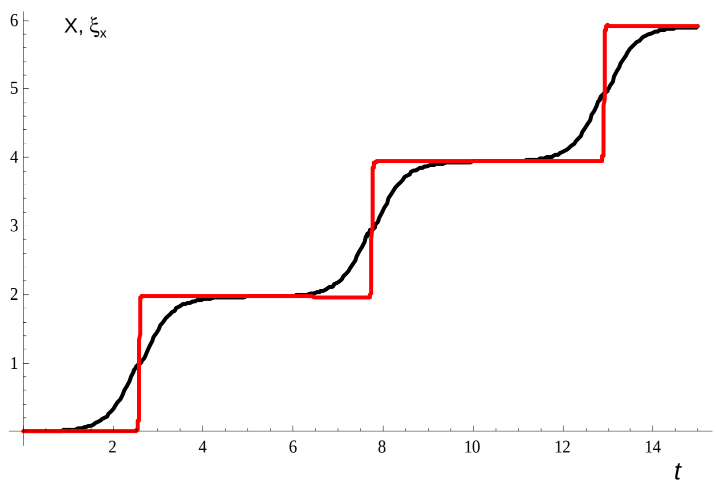


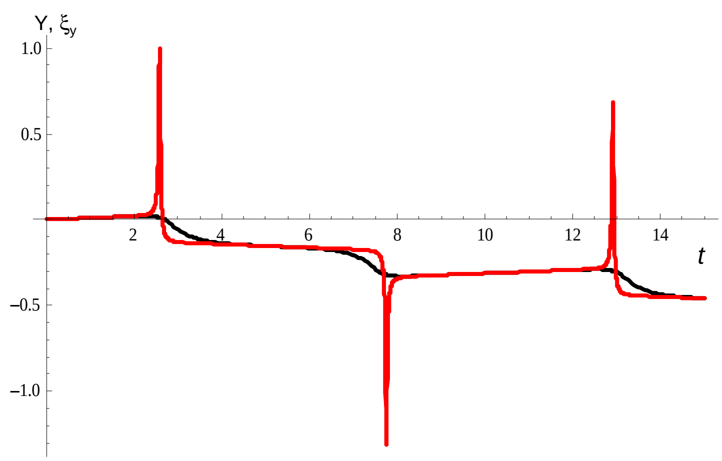




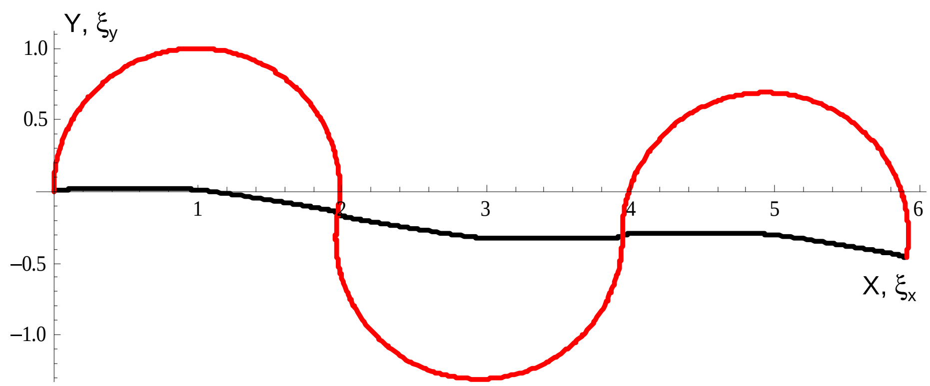
Let us now discuss the motion of CM and CP. The first figure shows $Z(t)$ that is given by Eq. (160) and could be figured out from the plot $\theta(t)$ above. The second figure shows $X(t)$ obtained by integration of Eq. (159) in thick blue and $\xi_x(t)$ defined by Eq. (163) in thin red. The third figure shows $Y(t)$ and $\xi_y(t)$ (note another vertical scale). One can see that as the wheel falls in the x direction and $X(t)$ increases, $\xi_x(t)$ remains nearly constant. However, as the wheel nearly hits the ground, $\xi_x(t)$ jumps forwards because of the fast rotation and reaches another constant value. In the third figure one can see wild excursions of $\xi_y(t)$ at nearly touch-the-ground moments.

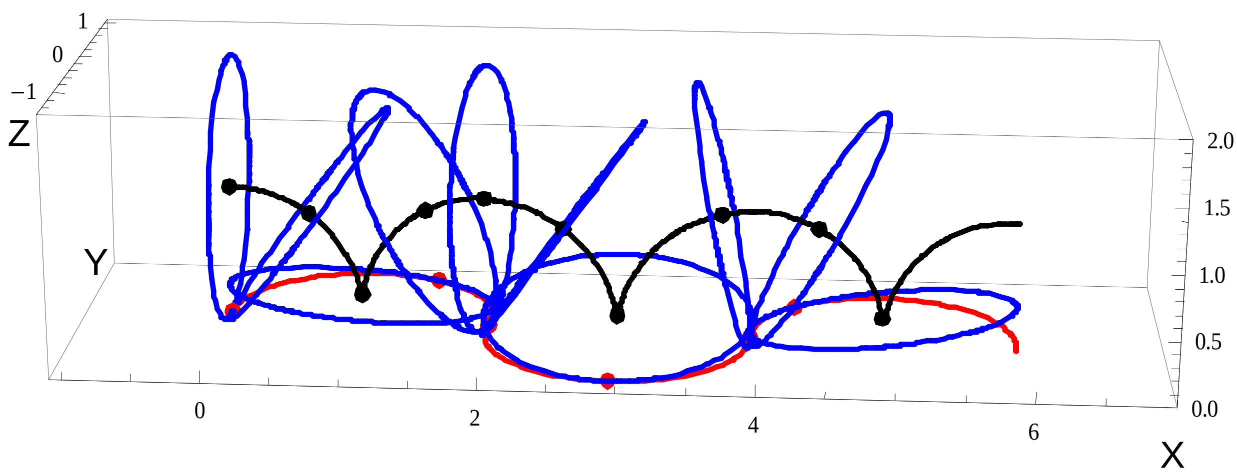




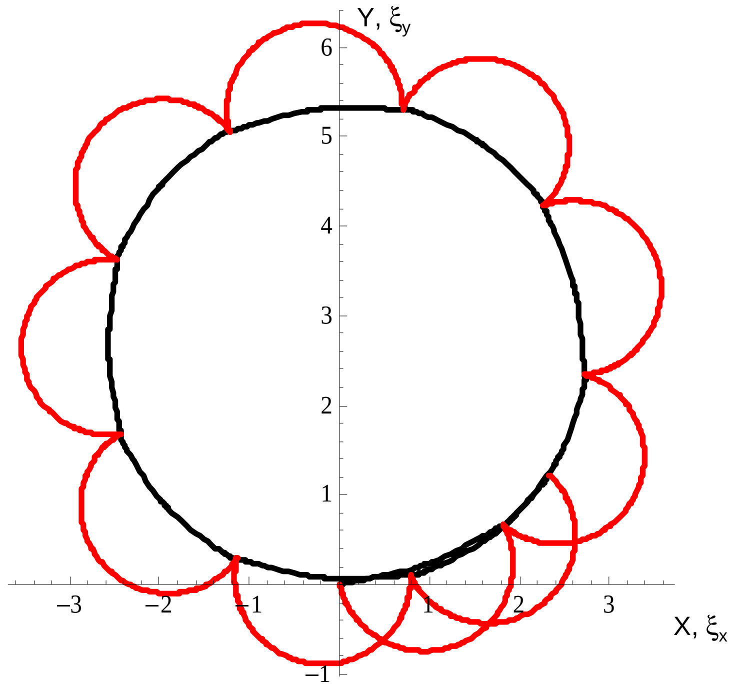


The figure below shows the trajectories of CM (projected on the plane) and CP. One can see that as the wheel nearly touches the ground CP makes a half-circle around CM. The next figure gives $3d$ representation of the trajectories and positions of the wheel at different moments of time.

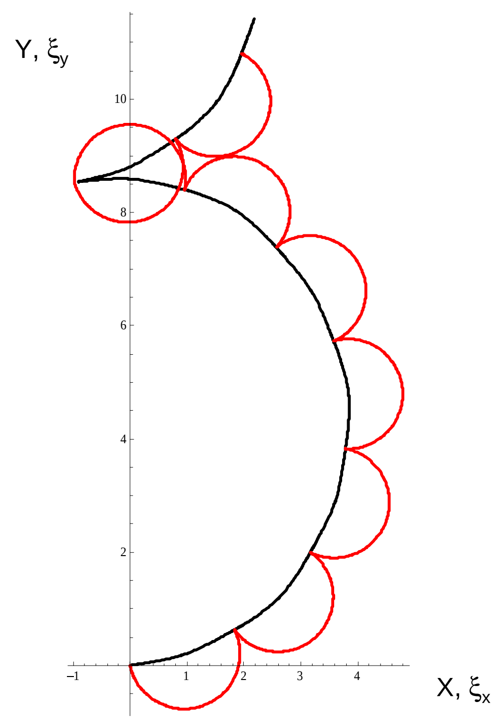




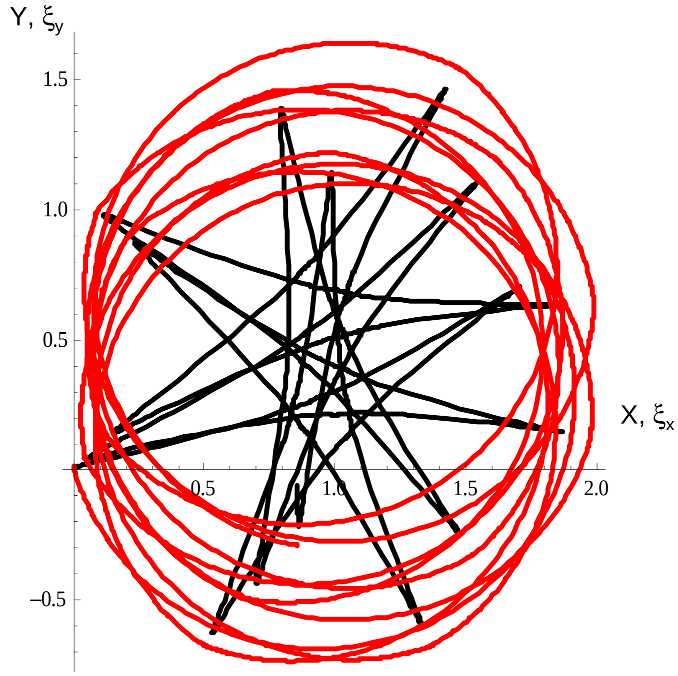
Initial precession causes the CM the wheel to make circles. The figure below shows the trajectories in the same case but with the initial precession $\dot{\phi}(0) = 0.1$.



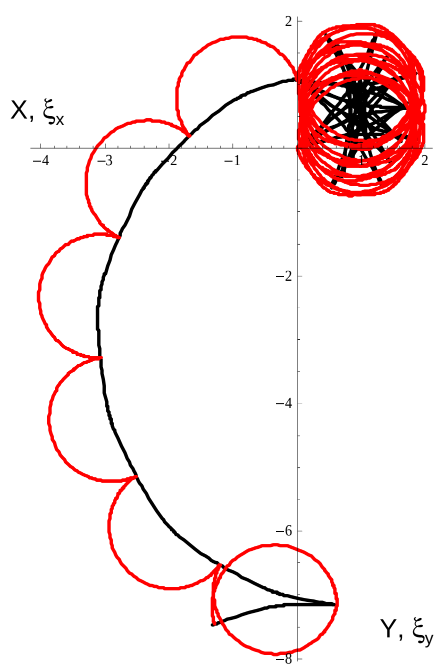
Adding a horizontal force, $\dot{\phi}(0) = 0.1$ and $F_x = 0.03$ in the figure below, makes the CM motion of a cycloid type.



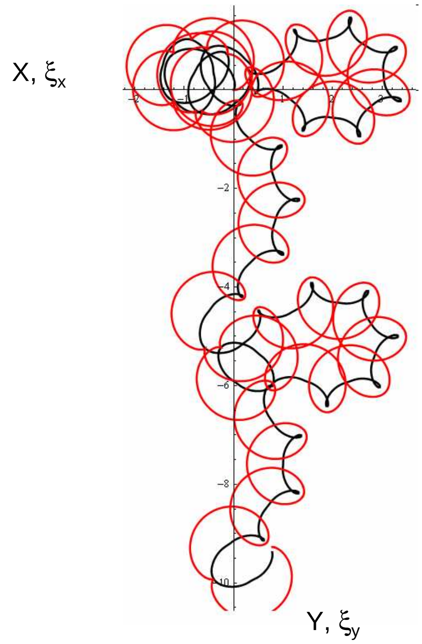
However, changing the sign of the force, $\dot{\phi}(0) = 0.1$ and $F_x = -0.03$ in the figure below, completely changes the character of the motion.



Below is the same with a longer time interval. One can see the change of regime after some time.



Faster precession makes the motion even more complicated, as below for $\dot{\phi}(0) = 1$ and $F_x = -0.03$.



In fact, in the nearly falling regime the average direction of the wheel's motion is quite unpredictable.

The problem also can be solved by initially formulating it with respect to CM. Here the only torque is that from the reaction force $\mathbf{F}^{(R)}$ applied to CP. The whole system of Newtonian equations of motion is given by

$$\begin{aligned} M\dot{\mathbf{V}} &= \mathbf{F} + \mathbf{F}^{(R)} \\ \dot{\mathbf{L}} &= \mathbf{K} = \left[-R\mathbf{e}^{(2)} \times \mathbf{F}^{(R)}\right], \end{aligned} \tag{194}$$

plus the constraint relation, Eq. (149). Eliminating $\mathbf{F}^{(R)}$ from the first equation,

$$\mathbf{F}^{(R)} = M\dot{\mathbf{V}} - \mathbf{F} = M\frac{d}{dt}\left(\left[\boldsymbol{\omega} \times R\mathbf{e}^{(2)}\right]\right) - \mathbf{F}, \tag{195}$$

and substituting the result into the second equation yields a closed equation of motion for $\dot{\mathbf{L}}$. After projecting onto the principal-axes frame $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$ and differentiating over time taking into account time dependence of $\mathbf{e}^{(\alpha)}$ (as was done in the derivation of the Euler equations) one arrrives at Euler equations with respect to the contact point, Eq. (151).

There is a subtle difference between the current problem of rotation around an instantaneous CP and rotation around a fixed point of support. In our case changing ψ does not change the orientation of the wheel, whereas for a fixed fulcrum at O' changing ψ makes the wheel cross the plane. In the absence of torques, the Euler equations for both models and their solutions are the same, if the embedded vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are used. Note that for the fixed-fulcrum model one cannot use the sliding vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ since the rotation ψ would make the tensor of inertia nondiagonal. The interpretations of the results is different, however. In particular, for the fixed-fulcrum model the unit vector $\mathbf{e}_O$ from O' to O is given by

$$\mathbf{e}_O = \mathbf{e}_A \cos\psi + \mathbf{e}_N \sin\psi. \tag{196}$$

Here changing ψ with time causes rotation of the center of mass around O'. In the case of applied torques the two models become non-equivalent. The torque with respect to O' is given by

$$\mathbf{K}' = \left[R\mathbf{e}_O \times \mathbf{F}\right], \tag{197}$$

whereas for our present model it is $\mathbf{K}' = [R\mathbf{e}_A \times \mathbf{F}]$, the first line of Eq. (153). One can see that formulation of the fixed-fulcrum model is more involved.