MINICOURSE: LIE-THEORETIC APPROACH TO THE BOCHNER TECHNIQUE

(joint work w/ Jackson Goodman) JULY 2023, MATRIX — CRESWICK, AUSTRALIA

PART 1

(M^n, g) Riem. mfld, oriented.

$Fr(M) = \{$orthonormal frames on $TM\}$ frame bundle

$SO(n) \to Fr(M) \to M$ principal bundle (can replace by princ. G-bundle if (M,g) has holonomy $G \subset SO(n)$.)

$$\boxed{\begin{array}{l}\pi : SO(n) \to SO(E) \\ \text{orthogonal/unitary representation}\end{array}} \rightsquigarrow \boxed{\begin{array}{l}E_\pi = Fr(M) \times_\pi E \longrightarrow M \\ \text{associated vector bundle}\end{array}}$$

$\pi_{\varepsilon_1} = \mathrm{id}$ (defining rep.) on $\mathbb{R}^n$ —— $E_{\pi_{\varepsilon_1}} = TM$ tangent bundle

$\pi^*_{\varepsilon_1}$ dual rep. on $(\mathbb{R}^n)^*$ —— TM^* cotangent bundle

(identify using g)

$\underbrace{\pi_{\varepsilon_1 + \dots + \varepsilon_p}}_{\omega_p}$ rep. on $\Lambda^p \mathbb{R}^n$, $p < \frac{n}{2}$ —— $\Lambda^p TM$ p-vectors / p-forms

p^{th} fund. weight.

$\pi_{p\varepsilon_1}$ rep. on $\mathrm{Sym}^p_0 \mathbb{R}^n$ —— $\mathrm{Sym}^p_0 TM$ traceless symm. p-tensors.

(This construction behaves nicely w.r.t. all linear algebra constructions:

$E_{\pi \oplus \pi'} = E_\pi \oplus E_{\pi'}$, $E_{\pi \otimes \pi'} = E_\pi \otimes E_{\pi'}$, $E_{\pi^*} = (E_\pi)^*$, $E_{\pi \otimes_{\mathbb{R}} \mathbb{C}} = E_\pi \otimes_{\mathbb{R}} \mathbb{C}$ ...)

$\pi_0 \oplus \pi_{2\varepsilon_1} \oplus \pi_{2\varepsilon_1 + 2\varepsilon_2}$ rep. on $\mathrm{Sym}^2_b(\Lambda^2 \mathbb{R}^n)$ —— $\mathrm{Sym}^2_b(\Lambda^2 TM)$ curvature operators.

$\cong$ (scal) $\oplus$ (traceless Ricci) $\oplus$ (Weyl) $(n \geq 5)$

[On $\mathfrak{so}(n)$, $n = 2m$ or $2m+1$, use o.n.b. $\varepsilon_i \in \mathfrak{t}^*$, $1 \leq i \leq m$, given by $\varepsilon_i\left(\mathrm{diag}\left(\begin{pmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \theta_m \\ -\theta_m & 0 \end{pmatrix}\right)\right) = \theta_i$.]

<u>Weitzenböck formula:</u> $\Delta = \nabla^* \nabla + t \cdot K(R, \pi)$ on sections of E_π.

$$\boxed{K(R,\pi) = -\sum_{a,b} R_{ab}\, d\pi(X_a) \circ d\pi(X_b) = -\sum_a d\pi(R X_a) \circ d\pi(X_a)}$$

$\pi : SO(n) \to SO(E)$

$d\pi : \mathfrak{so}(n) \to \mathfrak{so}(E)$

$\mathfrak{so}(n) \cong \Lambda^2 \mathbb{R}^n$

where $\{X_a\}$ is an o.n.b. of $\Lambda^2 T_p M \cong \mathfrak{so}(n)$ and $R = \sum_{a,b} R_{ab} X_a \otimes X_b$

is the curvature operator of (M^n, g).

Properties:

- $R \mapsto K(R,\pi)$ is $SO(n)$-equiv. linear map; $K(R,\pi)$ is self-adjoint
- $R \geq 0 \iff K(R,\pi) \geq 0,\ \forall \pi$ ($\Rightarrow$ trivial, $\Leftarrow$ [Hitchin, 2015])
- $\sec_R \geq 0 \iff K(R, \pi_{p\cdot\varepsilon_1}) \geq 0,\ \forall p \geq 2$ [B.–Mendes 2022]
- $K(R, \pi \oplus \pi') = K(R,\pi) \oplus K(R,\pi')$ is block-diagonal
- If π_λ is irred. rep. w/ highest weight λ

  then $K(\underset{\text{curvature operator of } (S^n, g_{round})}{Id}, \pi_\lambda) = d\pi_\lambda(-\mathrm{Cas}) \underset{\text{Freudenthal's formula}}{=} \langle \lambda, \lambda + 2\rho \rangle\, Id$

  ρ: half-sum of positive roots ("Weyl vector")

  e.g., on $SO(2m)$, $\rho = \sum_{i=1}^{m} (m-i)\varepsilon_i$

Bochner technique

If M is closed, and $\Delta\alpha = 0$, then

$$0 = \int_M \langle \Delta\alpha, \alpha \rangle = \int_M \|\nabla\alpha\|^2 + t\langle K(R,\pi)\alpha, \alpha \rangle$$

If $t \cdot K(R,\pi) \geq 0$, then $\nabla\alpha \equiv 0$ i.e. α is parallel

If $t \cdot K(R,\pi) > 0$, then $\alpha \in (E_\pi)_0$ subbundle corresponding to trivial isotypic component of π. (So $\alpha \equiv 0$ if π has no trivial factors! "Vanishing Theorem")

Ex: $K(R, \pi_{\varepsilon_1}) = \mathrm{Ric}$

On TM, with $t = -2$: $\Delta X = 0 \iff X$ is Killing vector field

On TM^*, with $t = 2$: $\Delta\alpha = 0 \iff \underbrace{(d\delta + \delta d)}\alpha = 0$ α harmonic 1-form

Hodge Laplacian on $\Lambda^p TM$ always corresponds to $t = 2$.

Bochner 1946: $\mathrm{Ric} < 0 \Rightarrow |\mathrm{Isom}(M^n, g)| < \infty$.

$\mathrm{Ric} > 0 \Rightarrow b_1(M, \mathbb{R}) = 0$

$\vdots$ (a lot in between that I'm skipping...)

[**Def:** R is k-positive if the sum of its smallest k eigenvalues is positive.]

Petersen–Wink 2021: R is $(n-p)$-positive $\Longrightarrow b_1(M) = \cdots = b_p(M) = 0$ ← All my Betti numbers are $b_k(M,\mathbb{R})$.

$1 \leq p \leq \frac{n}{2}$

$\left(b_{n-1}(M) = \cdots = b_{n-p}(M) = 0 \text{ by Poincaré duality} \right)$

In particular, R is $\frac{n}{2}$-pos. $\Longrightarrow M$ is a rational homology sphere.

Rmk: R is $(n-1)$-positive $\Longrightarrow$ $Ric > 0$ (but $\not\Leftarrow$).

R is 2-positive $\xRightarrow[\text{(Ricci flow)}]{[B.W.]}$ $M = S^n/\Gamma$ is a spherical space form

but 3-positivity etc are not preserved by Ricci flow

<u>Def</u>: If R has eigenvalues $\nu_1 \le \nu_2 \le \dots \le \nu_d$ and $0 < r \le d$, let

$$\Sigma(r, R) = \nu_1 + \dots + \nu_{\lfloor r \rfloor} + (r - \lfloor r \rfloor)\, \nu_{\lfloor r \rfloor + 1}$$

R is r-positive $\iff$ $\Sigma(r, R) > 0$ (for possibly noninteger values of r too)

<u>Proposition</u> (Key estimate). Let $\pi_\lambda : G \to SO(E)$ be a nontrivial irred. orthogonal/unitary G-representation w/ highest weight λ and $R : \mathfrak{g} \to \mathfrak{g}$ a self-adjoint operator. Then

$$K(R, \pi_\lambda) \geq \|\lambda\|^2 \cdot \Sigma\big(PW_G(\pi_\lambda), R\big) \cdot \mathrm{Id}$$

where $PW_G(\pi_\lambda) = \min\left\{ \dfrac{\langle \lambda, \lambda + 2\rho \rangle}{\|\lambda\|^2}, \dim \mathfrak{g} \right\}$.

similarly, we define the "Petersen-Wink invariant" on reducible G-repr. $\pi = \bigoplus_i \pi_i$ by setting $PW_G(\pi) = \min_i \{ PW_G(\pi_i) : \pi_i \text{ nontrivial} \}$

<u>Applications</u>:
- Lower bound for spectrum of Δ on any tensor bundle $E \to M$
- Vanishing theorems (Bochner technique) $\leftarrow$ will only discuss these here

<u>Fact</u>: If $\lambda = \sum_{j=1}^m a_j \varepsilon_j$, $n = 2m$ or $2m+1$, then $PW_{SO(n)}(\pi_\lambda) = 1 + \dfrac{\sum_j (n - 2j)\, a_j}{\sum_j a_j^2}$

Dominant Integral weights (by Highest Weight Theorem, these correspond to the irred. representations)

$a_j \in \mathbb{Z}$, $a_1 \geq a_2 \geq \dots \geq a_{m-1} \geq |a_m| \geq 0$ if $n = 2m$

$a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ if $n = 2m+1$

Thus: $PW_{SO(n)}(\underbrace{\varepsilon_1 + \dots + \varepsilon_p}_{\Lambda^p TM}) = n - p$ if $1 \le p \le \left\lfloor \frac{n}{2} \right\rfloor$ $\rightsquigarrow$ Vanishing Thm $b_1 = \dots = b_p = 0$. [Petersen-Wink, 2021]

$PW_{SO(n)}(\underbrace{p\,\varepsilon_1}_{\mathrm{Sym}^p_0 TM}) = \dfrac{n + p - 2}{p}$ if $1 \le p$.

Pf: Let $\{X_i\}$ be o.n.b that diagonalizes R, $R(X_i)=\nu_i X_i$, with $\nu_1 \le \nu_2 \le \cdots \le \nu_{\dim \mathfrak{g}}$.

Suppose $PW_G(\pi_\lambda) = \dfrac{\langle \lambda, \lambda+2\rho\rangle}{\|\lambda\|^2} < \dim \mathfrak{g}$, let $r = \lfloor PW_G(\pi_\lambda) \rfloor$.

Given $v \in E$, since π_λ is orthogonal/unitary, $d\pi_\lambda(X_i)$ is skew-adjoint:

$$\langle K(R,\pi_\lambda)v, v\rangle = \sum_{i=1}^{\dim \mathfrak{g}} \langle d\pi_\lambda(R(X_i))v,\ d\pi_\lambda(X_i)v\rangle$$

$$= \sum_{i=1}^{\dim \mathfrak{g}} \nu_i \|d\pi_\lambda(X_i)v\|^2$$

split into first/last r terms

$$\geq \sum_{i=1}^{r} \nu_i \|d\pi_\lambda(X_i)v\|^2 + \sum_{i=r+1}^{\dim \mathfrak{g}} \nu_{r+1} \|d\pi_\lambda(X_i)v\|^2$$

"add back" first r terms in second sum

$$= -\sum_{i=1}^{r} (\nu_{r+1}-\nu_i)\|d\pi_\lambda(X_i)v\|^2 + \nu_{r+1} \underbrace{\sum_{i=1}^{\dim \mathfrak{g}} \|d\pi_\lambda(X_i)v\|^2}_{\text{Casimir}}$$

$$\boxed{\|d\pi_\lambda(X)v\| \le \|\lambda\|\,\|v\|}$$

b/c up to conjugating may assume $X \in \mathfrak{t}^*$ is in Cartan subalgebra; eigenvalues of $d\pi_\lambda(X)$ are $\mu(X)$ for each weight $\mu \in \mathfrak{t}^*$ and $|\mu(X)| \underset{\text{Cauchy-Schwarz}}{\le} \|\mu\| \cdot \underbrace{\|X\|}_{=1} \le \|\lambda\|$

λ has max length among weights

$$\geq -\sum_{i=1}^{r} (\nu_{r+1}-\nu_i)\|\lambda\|^2\|v\|^2 + \nu_{r+1}\langle \lambda, \lambda+2\rho\rangle \|v\|^2$$

$$= \underbrace{\|\lambda\|^2 \left(\sum_{i=1}^{r} \nu_i + \left(PW_G(\pi_\lambda) - r\right)\nu_{r+1} \right)}_{\Sigma(PW_G(\pi_\lambda),\, R)} \|v\|^2$$

Suppose $PW_G(\pi_\lambda) = \dim \mathfrak{g} \le \dfrac{\langle \lambda, \lambda+2\rho\rangle}{\|\lambda\|^2}$. If $\nu_{\dim \mathfrak{g}} \ge 0$ similar argument works using $\nu_{\dim \mathfrak{g}}$ instead of ν_{r+1} in second sum; if $\nu_{\dim \mathfrak{g}} < 0$ then result follows from $\|d\pi(X_i)v\| \le \|\lambda\| \cdot \|v\|$. □

Rmk: The above lower bound is sharp: setting $R = \mathrm{Id}$, we have

$$K(\mathrm{Id}, \pi_\lambda) = \langle \lambda, \lambda + 2\rho \rangle \, \mathrm{Id} = \|\lambda\|^2 \cdot \Sigma(\mathrm{PW}_G(\pi_\lambda), R) \cdot \mathrm{Id}$$

Other vanishing theorems: For simplicity, we skip the "nonnegative curvature" version of the statements (but they also hold!)

- Vanishing thm for Hodge numbers of Kähler mflds [Petersen–Wink '21]

  (M^{2m}, g) closed Kähler mfld, $\phi \in \Lambda^{p,q} TM^*$ harmonic (p,q)-form $(1 \le p,q \le m)$

  Let $C^{p,q} := m + 1 - \frac{p^2+q^2}{p+q} \overset{\text{"Proof"}}{=} \mathrm{PW}_{U(m)}\left(\Lambda^{p,q}(\mathbb{C}^m)^*\right)$. not irreducible, so need to compare irred. parts!

  If $R: \mathfrak{u}(m) \to \mathfrak{u}(m)$ is $C^{p,q}$-positive, then $\phi \equiv 0$; in particular $h^{p,q}(M) = 0$.
  Hodge theory / Dolbeault coh.

- Tachibana-type thm (small improvement of [Petersen–Wink '21]):

  (M^n, g) closed Riem. mfld with harmonic curvature operator, $n \geqslant 5$ If $n=3,4$ then can replace $\frac{n-1}{2}$ with $\frac{n}{2}$.

  If R is $\frac{n-1}{2}$-positive, then (M^n, g) is a space form.

  Pf: $$\mathrm{PW}_{SO(n)}\big(\underbrace{\pi_0 \oplus \pi_{2\varepsilon_1} \oplus \pi_{2\varepsilon_1 + 2\varepsilon_2}}_{\pi}\big) = \begin{cases} \frac{n-1}{2} & \text{if } n \geqslant 5 \\ 2 & \text{if } n = 4 \\ 3/2 & \text{if } n = 3. \end{cases}, \quad t > 0.$$

  $E_\pi = \mathrm{Sym}^2_b(\Lambda^2 TM) \to \pi$ is bundle of curv. operators.

  Triv. isotypic comp. is $(E_\pi)_0 = \{ \kappa \cdot \mathrm{Id} : \kappa \in \mathbb{R} \}$ constant curvature curv. operators. $\square$

- Trace-free conformal Killing tensors

  (M^n, g) closed Riem. mfld, $\phi \in \mathrm{Sym}^p_0 TM$ harmonic. a.k.a. trace free conformal Killing tensor.

  If R is $\frac{n+p-2}{p}$-negative, then $\phi \equiv 0$.

  Pf: $\mathrm{PW}_{SO(n)}(\pi_{p \cdot \varepsilon_1}) = \frac{n+p-2}{p}$, $\quad t = -2 < 0$. $\square$

  cf. [Dairbekov–Sharafutdinov' 2010] and [Heil–Moroianu–Semmelmann' 2016] using instead $\sec < 0$.

- Twisted spinors: tomorrow! only other family of vanishing theorems with topological consequences besides those using Hodge Theory (use instead Atiyah–Singer Index Theorem).

PART 2:

(M^n, g) Riem. mfld, spin. (Stiefel-Whitney classes: $w_1 = 0$, $w_2 = 0$)

- Frame bundle $SO(n) \to Fr_{SO}(M) \to M$ lifts to principal $Spin(n)$-bundle
  $Spin(n) \to Fr_{Spin}(M) \to M$
- Associated bundle construction can be performed with spinor representations:

Yesterday: $\lambda = \sum_{j=1}^{m} a_j \varepsilon_j$, $a_j \in \mathbb{Z}$ (Irreducible $SO(2m)$-representations)

Dominant integral weights

$a_1 \geq a_2 \geq \cdots \geq a_{m-1} \geq |a_m| \geq 0$

$\cap$ ← can lift repr. by composing w/ double cover $Spin(2m) \to SO(2m)$

Today: $a_j \in \mathbb{Z}$ or $a_j + \frac{1}{2} \in \mathbb{Z}$. (Irreducible $Spin(2m)$-representations)

Def (PW-invariant): $PW_G(\pi_\lambda) = \dfrac{\langle \lambda, \lambda + 2\rho \rangle}{\|\lambda\|^2}$

same formula from yesterday holds b/c only depends on Lie algebra $\mathfrak{so}(n)$, which is the same for $SO(n)$ and $Spin(n)$.

For $G = SO(2m)$ or $Spin(2m)$, $n = 2m$, and $\lambda = \sum_{j=1}^{m} a_j \varepsilon_j$, $PW_G(\pi_\lambda) = 1 + \dfrac{\sum_j (n-2j) a_j}{\sum_j a_j^2}$

These are not lifts of $SO(2m)$-repr.

- Ex: $\begin{cases} \omega_{m-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{m-1} - \varepsilon_m) \\ \omega_m = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{m-1} + \varepsilon_m) \end{cases}$ $\rightsquigarrow$ $\pi_{\omega_{m-1}}$ and π_{ω_m} are the "half" spinor representations S^+ and S^-, which are irred. rep. of dim 2^{m-1}.

$\pi_S = \pi_{\omega_{m-1}} \oplus \pi_{\omega_m}$ "spinor repr.", $PW_{Spin(2m)}(\pi_{\omega_{m-1}}) = PW_{Spin(2m)}(\pi_{\omega_m}) = \underline{n-1}$. ⚠

so PW estimate in this case is much stronger to just imply $\underline{scal > 0}$

- (Complex) Spinor bundle: $S = S^+ \oplus S^- = E_{\pi_S} \to M$ vector bundle of rank 2^m

Dirac operator $D: S \to S$ is 1st order diff. operator s.t. $D^2 = \Delta$ satisfies Weitzenböck formula:

$D^2 = \nabla^* \nabla + t \cdot K(R, \pi_S)$, with $t = 2$ and $K(R, \pi_S) = \dfrac{scal}{8}$

can show this using formula for $K(R,\pi)$ from yesterday!

$\boxed{\text{Thm (Lichnerowicz '63). If } (M^n, g) \text{ is a closed spin Riem. mfld with scal} > 0\text{, then } \hat{A}(M) = 0.}$

Pf: (1) $D^2 = \nabla^* \nabla + \dfrac{scal}{4}$, so $scal > 0 \Rightarrow \operatorname{Ker} D = \{0\}$. (Vanishing theorem for harmonic spinors)

If $n = 4k$, then $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ w.r.t. $S = S^+ \oplus S^-$, thus $\operatorname{Ker} D^\pm = \{0\}$. (If $n \neq 4k$, then $\hat{A}(M) \equiv 0$.)

(2) Atiyah-Singer Index Theorem: $\hat{A}(M) = \operatorname{ind}(D^+) = \dim \operatorname{Ker} D^+ - \dim \underbrace{\operatorname{coker} D^+}_{= \operatorname{Ker} D^-} = 0$. □

Topological invariant: $\mathbb{Q}$-lin. comb. of Pontryagin numbers, so oriented cobordism invariant.

Example: K3 surface (Fermat quartic)

(Rmk: Need spin, e.g., $\hat{A}(\mathbb{C}P^2) = -\frac{1}{8}$ and has $scal > 0$.)

$M^4 = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{C}P^3 : x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}$ is spin and $\hat{A}(M^4) \neq 0$, so does not admit $scal > 0$.

Well-Known generalization: twisted spinors $E_{\pi_S \otimes \pi} = S \otimes E_\pi$ ($E_\pi \to M$ is a (cplx) vector bundle; e.g. $E_\pi = TM_{\mathbb{C}}$, $\Lambda^p TM_{\mathbb{C}}, \ldots$)

① Twisted Dirac operator $D_\pi : S\otimes E_\pi \longrightarrow S \otimes E_\pi$ satisfies Weitzenböck-type formula

$$D_\pi^2 = \nabla^*\nabla + \underbrace{\mathcal{R}_\pi}, \quad \text{where } \mathcal{R}_\pi \text{ depends on } R \text{ and } \pi.$$

This is no longer of the form $K(R,\pi)$!

Q: How to ensure $\mathcal{R}_\pi > 0$?

② Atiyah-Singer Index Theorem: $\hat{A}(M, E_\pi) = \langle \hat{A}(TM)\cdot ch(E_\pi), [M]\rangle = \operatorname{ind}(D^+_{E_\pi})$.

where $E_\pi \to M$ is complex vector bundle.

If E is "built from" TM, then $\hat{A}(M,E)$ is a rational linear comb. of Pontryagin numbers of M, just like $\hat{A}(M) = \langle \hat{A}(TM), [M]\rangle$ and R_E depends only on R.

A:

$$\boxed{\text{Prop: } \mathcal{R}_\pi = K(R, \pi_S\otimes\pi) + \frac{\mathrm{scal}}{8}\,\mathrm{Id} - K(R,\pi), \quad \text{where } D_\pi^2 = \nabla^*\nabla + \mathcal{R}_\pi}$$

Pf: $d\pi_S(e_i\wedge e_j) = \frac{1}{2}\underbrace{e_i e_j}_{\text{Clifford mult.}}$ and simple (but important!) computation that I'll skip here to handle the mixed terms from $d(\pi_S\otimes\pi) = d\pi_S\otimes 1 + 1\otimes d\pi$ multiplied by $d\pi$.

From yesterday: $\boxed{K(R,\pi) \geq \|\lambda\|^2 \cdot \Sigma(PW_G(\pi_\lambda), R)\cdot \mathrm{Id}}$

Example: $\pi = \pi_{\varepsilon_1}$, $\quad S\otimes E_\pi = S\otimes TM_{\mathbb{C}}$

"Rarita-Schwinger fields" (spin $\frac{3}{2}$ fermions)

Pieri's formula (Littlewood-Richardson rule) $\Rightarrow \begin{cases} \pi_{\omega_m}\otimes\pi_{\varepsilon_1} = \pi_{\omega_m+\varepsilon_1} \oplus \pi_{\omega_{m-1}} \\ \pi_{\omega_{m-1}}\otimes\pi_{\varepsilon_1} = \pi_{\omega_{m-1}+\varepsilon_1}\oplus\pi_{\omega_m}\end{cases} \Rightarrow S\otimes E_\pi = S \oplus \overbrace{E_{\pi_{\omega_m+\varepsilon_1}} \oplus E_{\pi_{\omega_{m-1}+\varepsilon_1}}}$

$$PW_{Spin(2m)}(\pi_{\omega_m}\otimes\pi_{\varepsilon_1}) = PW_{Spin(2m)}(\pi_{\omega_{m-1}}\otimes\pi_{\varepsilon_1}) = \min\Big\{\underbrace{PW(\pi_{\omega_m})}_{= n-1}, \underbrace{PW(\pi_{\omega_m+\varepsilon_1})}_{= \frac{n(n+7)}{n+16}}\Big\} = \frac{n(n+7)}{n+16}$$

from now on I just write $PW = PW_{Spin(2m)}$

So

$$\mathcal{R}_{\pi_{\varepsilon_1}} = K(R, \pi_S\otimes\pi_{\varepsilon_1}) + \frac{\mathrm{scal}}{8}\mathrm{Id} - \mathrm{Ric} \qquad (\text{recall } K(R,\pi_{\varepsilon_1}) = \mathrm{Ric})$$

$$\geq \begin{cases} \left(\underbrace{\|\omega_m+\varepsilon_1\|^2}_{= (\frac{n}{8}+2)}\Sigma\Big(\underbrace{\frac{n(n+7)}{n+16}}_{= r_1}, R\Big) + \frac{\mathrm{scal}}{8} - \mu\right)\mathrm{Id} \text{ on } E_{\pi_{\omega_{m-1}+\varepsilon_1}}\oplus E_{\pi_{\omega_m+\varepsilon_1}} \\ \left(\frac{\mathrm{scal}}{8} + \frac{\mathrm{scal}}{8} - \mu\right)\mathrm{Id} \text{ on } S = E_{\pi_{\omega_m}}\oplus E_{\pi_{\omega_{m-1}}}\end{cases}$$

$$\geq C_1(R)\cdot\mathrm{Id}, \quad \text{where } C_1(R) = \min\left\{\left(\frac{n}{8}+2\right)\Sigma(r_1, R), \frac{\mathrm{scal}}{8}\right\} + \frac{\mathrm{scal}}{8} - \mu$$

μ = largest eigenvalue of Ric. ← or $\Sigma(n-1, R)$ if one wishes to use only Spec(R)...

$\boxed{\text{Thm. If } (M^n, g) \text{ is a closed Riem. spin mfld with } C_1(R) > 0, \text{ then } \hat{A}(M, TM_{\mathbb{C}}) = 0.}$

$n = 4k,\ k \geq 2$

Example. $M^8 = \mathbb{H}P^2$ does not admit a metric with $C_1(R) > 0$, since $\hat{A}(M, TM_{\mathbb{C}}) \neq 0$.

same for $\#^k \mathbb{H}P^2$, $k \geq 1$

Cor $(n=8)$. $r_1 = 5$ so $C_1(R) = \min\left\{3(\nu_1 + \cdots + \nu_5), \frac{\mathrm{scal}}{8}\right\} + \underbrace{\frac{\mathrm{scal}}{8} - \mu}_{=0 \text{ if } R \text{ is Einstein w/ scal}>0}$, and thus a spin closed mfld M^8 w/ $\hat{A}(M, TM_{\mathbb{C}}) \neq 0$ does not admit Einstein metrics w/ 5-positive R.

Note: Fubini–Study metric on $\mathbb{H}P^2$ is 19-positive.

General case:

$\forall p \geq 1$, let $C_p(R) = \min\left\{\left(\frac{n}{8} + p^2 + p\right)\Sigma(r_p, R), \frac{n(n-8)}{8 r_p}\Sigma(r_p, R)\right\} + \frac{\mathrm{scal}}{8} + p^2\, \Sigma(r_p', -R)$

⚠ this is a sum of largest r_p' eigenvalues of R.

where $r_p, r_p' \in \mathbb{Q}$ are explicit constants that depend on n.

$$\left(r_p = \frac{n^2 + (8p-1)n + 8p(p-1)}{n + 8p(p+1)} \qquad r_p' = \frac{n+p-2}{p} \right)$$

Note:
- $C_p(R)$ is a (piecewise) linear comb. of eigenvalues of R for $p \geq 2$.
  b/c of "min"   e-val's of R and Ric if $p=1$
- If $1 \leq q < p$, then $C_p(R) > 0 \Rightarrow C_q(R) > 0 \Rightarrow \mathrm{scal} > 0$.
- $\{R : C_p(R) \geq 0\}$ is a spectrahedron.

Thm A (B.–Goodman '22). If (M^n, g), $n = 4k \geq 8$, is a closed Riem. spin mfld with $C_p(R) > 0$, and $E \subseteq TM^{\otimes p}$ is a parallel subbundle, then $\hat{A}(M, E_{\mathbb{C}}) = 0$. ← rational cobordism invariant!

Pf: For all irred. $SO(n)$-rep. π_λ which is a subrepresentation of $\pi_{\varepsilon_1}{}^{\otimes p}$, we have

$$\begin{cases} PW(\pi_S \otimes \pi_\lambda) \geq PW(\pi_S \otimes \pi_{p \cdot \varepsilon_1}) =: r_p \\ PW(\pi_\lambda) \geq PW(\pi_{p\cdot\varepsilon_1}) =: r_p' \end{cases}$$

and similarly for $\langle \lambda, \lambda + 2\rho \rangle$ compared to $\langle p\cdot\varepsilon_1, p\cdot\varepsilon_1 + 2\rho\rangle$, i.e., "worst case scenario" happens if $\lambda = p\cdot\varepsilon_1$ (this is shown using convex optimization!)

and $r \mapsto \Sigma(r, R)/r$ is $\nearrow$, so $C_p(R) > 0 \Rightarrow R_{\pi_\lambda} > 0 \xRightarrow{\text{Atiyah–Singer}} \hat{A}(M, (E_{\pi_\lambda})_{\mathbb{C}}) = 0$ □

Example: $M^{16} = \mathbb{C}a P^2$ does not admit a metric w/ $C_2(R) > 0$, since $\hat{A}(M, \Lambda^2 TM_{\mathbb{C}}) \neq 0$.

Remark: Specializing to a given parallel subbundle $E \subseteq TM^{\otimes p}$, such as $\Lambda^p TM$ or $\mathrm{Sym}_0^p TM$, it is possible to obtain better estimates (weaker curvature requirements) in order to prove $\hat{A}(M, E_{\mathbb{C}}) = 0$.

Q: what manifolds have $C_p(R) > 0$? / Are we obstructing an impossible curvature condition?!

A: "Lots" of manifolds have $C_p(R)>0$:

Thm B. (B. – Goodman '22).

(i) Every nontorsion cobordism class in Ω_n^{SO}, $n\geq 10$, contains a manifold with $C_1(R)>0$.

i.e., without spin condition, there is no restriction on rational cobordism class.

(ii) If M^n is spin, $n\geq 10$, and $\hat{A}(M)=\hat{A}(M,TM_{\mathbb{C}})=0$, then $\#^\ell M^n$ is spin-cobordant to a mfld with $C_1(R)>0$.

i.e., with spin condition, these are the only restrictions on the rational cobordism class.

(iii) $C_p(R)>0$ is preserved under surgeries* of codimension d if $(d-1)(d-3)>8p(p+n-2)$.

Note: Round spheres S^n have $C_p(\mathrm{Id})>0$ if $n\gg p$

$$C_p(\mathrm{Id})=\frac{1}{4}n^2-\left(p+\frac{1}{4}\right)n-p(p-2)$$

but, of course, $M=S^n=\partial B^{n+1}$ so $\hat{A}(M,E_{\mathbb{C}})=0$ for any E.

*Recall: Surgery of codimension d is to
- Remove $S^{n-d}\times D^d\subset M^n$
- Glue in $D^{n-d+1}\times S^{d-1}$

Result is cobordant but has
- smaller b_{n-d} if $0\neq[S^{n-d}]\in H_{n-d}(M,\mathbb{Q})$
- larger b_{n-d+1} if $0=[S^{n-d}]\in H_{n-d}(M,\mathbb{Q})$

cf. Gromov-Lawson / Schoen-Yau: $\mathrm{scal}>0$ is preserved if $d\geq 3$.

Applications to other cobordism invariants:

- Cobordism class itself:

Thm C (B. – Goodman '22). Let (M^{4k},g) be a closed Riem. spin manifold, with $\frac{\mathrm{scal}}{8}\mathrm{Id}-\mathrm{Ric}>0$ and

$$\left.\begin{array}{ll}\Sigma(5,R)>0 & \text{if } k=2,\\ \Sigma(2k+4,R)>0 & \text{if } k\geq 6 \text{ even},\\ \Sigma(2k+6)>0 & \text{if } k\geq 9 \text{ odd},\end{array}\right\}$$

Note: these conditions are always weaker than $\Sigma(\lceil\frac{n}{2}\rceil,R)=\Sigma(2k,R)>0$; cf. Petersen – Wink.

then M is rationally null-cobordant, i.e., $\#^\ell M^n=\partial W^{n+1}$ for some $\ell\geq 1$.

Method of proof: [Thom' 1954]: Pontryagin numbers $(p_{I_1},\dots,p_{I_{p(k)}})\colon \Omega_{4k}^{Spin}\otimes\mathbb{Q}\cong\Omega_{4k}^{SO}\otimes\mathbb{Q}\xrightarrow{\cong}\mathbb{Q}^{p(k)}$ ← partitions of k

define an isomorphism, so M is rationally null-cobordant iff all its Pontryagin numbers vanish.

- So it suffices to show that $\hat{A}(M,E_{\mathbb{C}})=\begin{pmatrix}\mathbb{Q}\text{-linear combination}\\ \text{of Pontryagin numbers}\end{pmatrix}=0$ for sufficiently many E's

  coefficients depend on choice of E; e.g.:

- Achieve this applying Thm A and Petersen–Wink's result.

$\Omega_*^{SO}\otimes\mathbb{Q}\cong\mathbb{Q}[\mathbb{CP}^2,\mathbb{CP}^4,\dots]$

$\Omega_*^{SO}/\mathrm{torsion}\cong\mathbb{Q}[\mathbb{CP}^{2m},H_{ij}:{\scriptstyle m\geq 1,\ i\geq 2,\ j\geq 6}]$

$\Omega_*^{Spin}\otimes\mathbb{Q}\cong\mathbb{Q}[K3,\mathbb{HP}^2,\mathbb{HP}^3,\dots]$

$n=4$: p_1	$n=8$: p_1^2, p_2	$n=12$: p_1^3, p_1p_2, p_3	...
$\hat{A}(M^4)=-\frac{p_1}{24}$	$\hat{A}(M^8)=\frac{7p_1^2-4p_2}{5760}$	$\hat{A}(M^{12})=\frac{-31p_1^3+44p_1p_2-16p_3}{967680}$	
$\hat{A}(M^4,TM_{\mathbb{C}})=\frac{5p_1}{6}$	$\hat{A}(M^8,TM_{\mathbb{C}})=\frac{37p_1^2-124p_2}{720}$	$\hat{A}(M^{12},TM_{\mathbb{C}})=\frac{11p_1^3-124p_1p_2+656p_3}{80640}$	
$\hat{A}(M^4,\mathrm{Sym}^2TM_{\mathbb{C}})=\frac{67p_1}{12}$	$\hat{A}(M^8,\mathrm{Sym}^2TM_{\mathbb{C}})=\frac{701p_1^2-1292p_2}{480}$	$\hat{A}(M^{12},\mathrm{Sym}^2TM_{\mathbb{C}})=\frac{20933p_1^3-64612p_1p_2+58928p_3}{161280}$	
$\hat{A}(M^4,\Lambda^2TM_{\mathbb{C}})=\frac{7p_1}{4}$	$\hat{A}(M^8,\Lambda^2TM_{\mathbb{C}})=\frac{409p_1^2-28p_2}{1440}$	$\hat{A}(M^{12},\Lambda^2TM_{\mathbb{C}})=\frac{499p_1^3+3844p_1p_2-27056p_3}{161280}$	

Remark: If $n=8$, then $\Omega_8^{SO}\cong\mathbb{Z}\oplus\mathbb{Z}$ has no torsion, so $\frac{\mathrm{scal}}{8}-\mathrm{Ric}>0$ and $\Sigma(5,R)>0$ actually implies M^8 is null-cobordant.

- Witten genus: $\varphi_W(M) = \hat{A}\left(M, \bigotimes_{\ell=1}^{\infty} \mathrm{Sym}_{q^\ell} TM_{\mathbb{C}}\right) \prod_{\ell=1}^{\infty} (1-q^\ell)^{4k}$, where $\mathrm{Sym}_t TM_{\mathbb{C}} = \mathbb{C} + TM_{\mathbb{C}}\, t + \mathrm{Sym}^2 TM_{\mathbb{C}}\, t^2 + \dots$

is a formal power series $\varphi_W(M) \in \mathbb{Q}[[q]]$ with coefficients $\hat{A}(M)$, $\hat{A}(M, TM_{\mathbb{C}})$, $\hat{A}(M, TM_{\mathbb{C}} \oplus \mathrm{Sym}^2 TM_{\mathbb{C}})$ ...

Witten: $\varphi_M(M)$ "is" the S^1-equivariant index of Dirac operator on loop space ΩM. ← not rigorously defined

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Thm D. (B.–Goodman '22) Let (M^{4k}, g) be a closed Riem. spin manifold, and set

$$p = \left\lfloor \frac{k}{6} \right\rfloor - 1 \quad \text{if } k \equiv 1 \bmod 6, \qquad p = \left\lfloor \frac{k}{6} \right\rfloor \text{ otherwise} \qquad \left(\text{so } p \approx \frac{\dim M}{24}.\right)$$

If $p \geq 1$, $C_p(R) > 0$, and $p_1(TM) = 0$, then $\varphi_W(M) = 0$.

Conjecture (Stolz, 1996). If (M^n, g) has $\mathrm{Ric} > 0$ and $\underbrace{\tfrac{1}{2} p_1(TM) = 0}_{\text{"string"}}$, then $\varphi_W(M) = 0$. ← if true, would yield first examples of simply-connected manifolds with $\mathrm{scal} > 0$ that do not admit $\mathrm{Ric} > 0$.

Note: $\mathrm{Ric} > 0$ does not imply $C_p(R) > 0$!
There exist manifolds with $C_p(R) > 0$ with p as above and $|\pi_1 M| = +\infty$, so no $\mathrm{Ric} > 0$.

Method of proof:
- $p_1(TM) = 0 \Rightarrow \varphi_W(M)$ is a modular form of weight $2k$

  $\Rightarrow \varphi_W(M) = 0$ if first $p \approx \frac{k}{6}$ coeff. vanish.
- Apply Theorem A to show these coeff. vanish, since they involve $\hat{A}(M, E_{\mathbb{C}})$, $E \subseteq TM^{\otimes p}$. $\square$

- Elliptic genus: $$\phi(M) = \left(2 \prod_{\ell=1}^{\infty} \frac{(1-q^\ell)^2}{(1+q^\ell)^2}\right)^{2k} \left\langle L(TM) \cdot \mathrm{ch}\left(\psi_2\left(\bigotimes_{\ell=1}^{\infty} \mathrm{Sym}_{q^\ell} TM_{\mathbb{C}} \otimes \Lambda_{q^\ell} TM_{\mathbb{C}}\right)\right), [M] \right\rangle,$$

So first term is signature $\sigma(M) = \langle L(TM), [M] \rangle$ — Adams operation — analogous to Sym_{q^ℓ} for Λ^p.

Thm E. (B.–Goodman '22). Let (M^{4k}, g) be a closed Riem. spin mfld. If $k \geq 2$ and $C_{\lfloor k/2 \rfloor}(R) > 0$, then the elliptic genus $\phi(M)$ vanishes (and hence the signature $\sigma(M)$ vanishes).