Counting homogeneous Einstein metrics

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Homogeneous Einstein metrics

- (M^n, g) compact Riemannian manifold, $p \in M$
- $\mathsf{G} \subset \mathsf{Iso}(M^n, \mathrm{g})$ compact Lie group, acts transitively
- $\mathsf{H} = \{g \in \mathsf{G} : g \cdot p = p\}$ isotropy, so $M \cong \mathsf{G}/\mathsf{H}$
- Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, decompose $\mathfrak{m}$ into H-irreducibles

$$T_pM \;\cong\; \mathfrak{m} \;=\; \underline{\quad \mathfrak{m}_1 \;\oplus\; \dots \;\oplus\; \mathfrak{m}_\ell \quad}$$

- Assume $\underline{\;\mathfrak{m}_i \;\not\cong\; \mathfrak{m}_j, \text{ for all } i \neq j\;}$
- Then, all G-invariant Riemannian metrics g are of the form

$$\mathrm{g}_p \colon T_pM \to T_pM, \quad \mathrm{g}_p \;=\; \underline{x_1\, Q|_{\mathfrak{m}_1} + \cdots + x_\ell\, Q|_{\mathfrak{m}_\ell}}\,,$$

where $x_i > 0$ and Q is a fixed bi-invariant metric on G.

Ricci tensor of $g_p = x_1\, Q|_{\mathfrak{m}_1} + \cdots + x_\ell\, Q|_{\mathfrak{m}_\ell}$ is $(\operatorname{Ric}_g)_p \colon T_pM \to T_pM$,

$$(\operatorname{Ric}_g)_p = r_1(\mathbf{x})\, x_1\, Q|_{\mathfrak{m}_1} + \cdots + r_\ell(\mathbf{x})\, x_\ell\, Q|_{\mathfrak{m}_\ell} \qquad \mathbf{x} = (x_1, \dots, x_\ell)$$

$$= r_1(\mathbf{x})\, g_p|_{\mathfrak{m}_1} + \cdots + r_\ell(\mathbf{x})\, g_p|_{\mathfrak{m}_\ell}$$

$$\boxed{r_i(\mathbf{x}) = \frac{b_i}{2x_i} - \frac{1}{4d_i} \sum_{j,k=1}^{\ell} L_{ijk}\, \frac{2\,x_k^2 - x_i^2}{x_i\, x_j\, x_k}}$$

Laurent polynomials with "parameters" $b_i \geq 0,\ d_i > 0,\ L_{ijk} \geq 0,$

$$B|_{\mathfrak{m}_i} = -b_i\, Q|_{\mathfrak{m}_i}, \quad d_i := \dim \mathfrak{m}_i, \quad L_{ijk} := \sum_{\substack{\mathbf{v}_\alpha \in \mathfrak{m}_i,\, \mathbf{v}_\beta \in \mathfrak{m}_j \\ \mathbf{v}_\gamma \in \mathfrak{m}_k}} Q([\mathbf{v}_\alpha, \mathbf{v}_\beta], \mathbf{v}_\gamma)^2$$

L_{ijk} is fully symmetric! $\mathbf{v}$'s are Q-orthonormal

$\mathbf{b} = (b_i)$, $\mathbf{d} = (d_i)$, $L = (L_{ijk})$ depend on G/H and $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\ell$

Einstein: $\boxed{\operatorname{Ric}_g = \cancelto{1}{c}\; g \iff r_i(\mathbf{x}) = \cancelto{1}{c},\ \forall i}$

Natural questions

Existence

Does $r_1(\mathbf{x}) = \cdots = r_\ell(\mathbf{x}) = 1$ admit solutions $\mathbf{x} \in \mathbb{R}^\ell_+$?

Theorem (Wang–Ziller, 1986)

- *Yes, if* $\mathfrak{h} \subset \mathfrak{g}$ *is* <u>*maximal Lie subalgebra*</u>
- *No, e.g., on* G/H, *where* <u>$\mathsf{H} = \mathsf{SU}(2) \subset \mathsf{Sp}(2) \subset \mathsf{SU}(4) = \mathsf{G}$</u>

And a lot more: Böhm, Böhm–Wang–Ziller, Dickinson–Kerr, ...

Classification

Classify all solutions to $r_1(\mathbf{x}) = \cdots = r_\ell(\mathbf{x}) = 1$.

Known, e.g., if G/H <u>is a CROSS, and a few other special cases</u>

Finiteness Conjecture (Böhm–Wang–Ziller, 2004)

There are **finitely many** solutions to $r_1(\mathbf{x}) = \cdots = r_\ell(\mathbf{x}) = 1$.

Main results (Geometric version)

Theorem

If G/H *is a compact homogeneous space whose isotropy representation* $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\ell$ *consists of* ℓ *pairwise inequivalent irreducible summands, then there are at most*

$$D_{\ell-1} = \sum_{k=0}^{\ell-1} 2^k \binom{\ell-1}{k}^2$$

isolated G*-invariant Einstein metrics* g *on* G/H *with* $\mathrm{Ric}_{\mathrm{g}} = \mathrm{g}$.

All G*-invariant Einstein metrics with* $\mathrm{Ric}_{\mathrm{g}} = \mathrm{g}$ *are isolated if* $E_A(\mathrm{scal}) \neq 0$*, so the Finiteness Conjecture holds in such cases.*

"Central Delannoy numbers"

$D_1 = 3,\quad D_2 = 13,\quad D_3 = 63,\quad D_4 = 321,$

$D_5 = 1\,683,\quad D_6 = 8\,989,\quad D_7 = 48\,639,\quad D_8 = 265\,729,\quad \ldots$

Main results (Algebraic version)

Theorem

(i) *For all* $\mathbf{b}$, $\mathbf{d}$, *and* L, *there are at most* $D_{\ell-1}$ *isolated solutions* $\mathbf{x} \in (\mathbb{C}^*)^\ell$ *to the Einstein equations* $r_1(\mathbf{x}) = \cdots = r_\ell(\mathbf{x}) = 1$.

(ii) *For generic* $\mathbf{b}$, $\mathbf{d}$, *and* L, *all solutions to the Einstein equations are isolated and there are exactly* $D_{\ell-1}$ *solutions in* $(\mathbb{C}^*)^\ell$.

(iii) *For each support* A, *there is a polynomial* $E_A(\text{scal})$ *on* $\mathbf{b}$, $\mathbf{d}$, *and* L *such that* $E_A(\text{scal}) \neq 0$ *implies* $\mathbf{b}$, $\mathbf{d}$, *and* L *are generic.*

$$r_i(\mathbf{x}) = \frac{b_i}{2x_i} - \frac{1}{4d_i} \sum_{j,k=1}^{\ell} L_{ijk} \frac{2\,x_k^2 - x_i^2}{x_i\,x_j\,x_k}, \quad i = 1, \ldots, \ell$$

Support and Newton polytope

$$\mathbf{x} = (x_1, \ldots, x_\ell) \in (\mathbb{C}^*)^\ell$$
$$\mathbf{a} = (a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell$$
$$\mathbf{x}^\mathbf{a} := x_1^{a_1} \ldots x_\ell^{a_\ell}$$

Define, for a Laurent polynomial

$$f(\mathbf{x}) = \sum_{\mathbf{a}\in\mathbb{Z}^\ell} c_\mathbf{a}\, \mathbf{x}^\mathbf{a},$$

- Support:

  $\operatorname{supp} f := \{\mathbf{a} \in \mathbb{Z}^\ell : c_\mathbf{a} \neq 0\}$

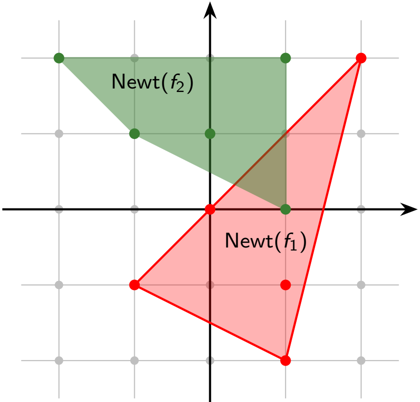
- Newton polytope:

  $\operatorname{Newt}(f) := \operatorname{conv}(\operatorname{supp} f)$

Example

$$f_1(\mathbf{x}) = \frac{\sqrt{5}}{x_1 x_2} + 3\frac{x_1}{x_2^2} - \frac{x_1}{x_2} + 1 - x_1^2 x_2^2$$

$$f_2(\mathbf{x}) = \frac{x_2^2}{x_1^2} + \frac{x_2}{x_1} - x_1 + 8x_2 + x_1 x_2^2$$



Mixed volume

Given $P_1, \ldots, P_\ell \subset \mathbb{R}^\ell$ polytopes, $\lambda_1, \ldots, \lambda_\ell > 0$, the volume of

$$\lambda_1 P_1 + \cdots + \lambda_\ell P_\ell := \underline{\left\{ \sum_{j=1}^{\ell} \lambda_j \mathbf{p}_j : \mathbf{p}_j \in P_j \right\}}$$

is a homogeneous polynomial $V(\lambda_1, \ldots, \lambda_\ell)$ of degree ℓ.

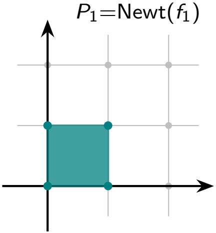
Definition (Mixed volume)

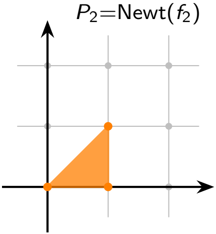
$\mathrm{MV}(P_1, \ldots, P_\ell)$ is the coefficient of $\lambda_1 \ldots \lambda_\ell$ in $V(\lambda_1, \ldots, \lambda_\ell)$.

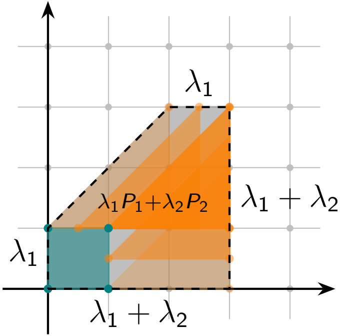
- P_j lattice polytopes $\implies \mathrm{MV}(P_1, \ldots, P_\ell)$ is an integer.
- $\mathrm{MV}(P_1, \ldots, P_\ell)$ is the unique symmetric multilinear function such that $\mathrm{MV}(P, \ldots, P) = \ell!\ \mathrm{Vol}(P)$.

Example

$$f_1(\mathbf{x}) = 1 - x_1 - x_2 + x_1\, x_2 \qquad f_2(\mathbf{x}) = 1 - t\,x_1 + x_1\,x_2, \quad t \neq 0$$







$$V(\lambda_1, \lambda_2) = (\lambda_1 + \lambda_2)^2 - \tfrac{1}{2}\lambda_2^2$$

$$V(\lambda_1, \lambda_2) = \lambda_1^2 + 2\,\lambda_1\lambda_2 + \tfrac{1}{2}\lambda_2^2$$

$$\mathrm{MV}(P_1, P_2) = 2$$

Bernstein's Theorem

- $\mathcal{F} = \{f_1, \ldots, f_\ell \colon (\mathbb{C}^*)^\ell \to \mathbb{C}\}$ system of Laurent polynomials
- $\mathbf{x} \in (\mathbb{C}^*)^\ell$ *solution* to $\mathcal{F} \iff f_1(\mathbf{x}) = \cdots = f_\ell(\mathbf{x}) = 0$

Bernstein–Khovanskii–Kushnirenko

Bernstein's Theorem (1975), "BKK bound"

The system $\mathcal{F} = \{f_1, \ldots, f_\ell\}$ of Laurent polynomials has at most $\mathrm{MV}(P_1, \ldots, P_\ell)$ isolated solutions in $(\mathbb{C}^*)^\ell$, where $P_j = \mathrm{Newt}\,(f_j)$.

Example

$f_1(\mathbf{x}) = 1 - x_1 - x_2 + x_1 x_2, \quad f_2(\mathbf{x}) = 1 - t\, x_1 + x_1 x_2,$

$\mathrm{MV}(P_1, P_2) = \underline{\ 2\ }$, provided $t \neq 0$.

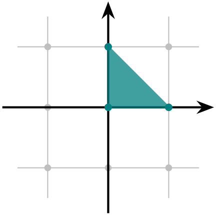
Solutions to $\mathcal{F} = \{f_1, f_2\}$: $\mathbf{x} = \underline{(1,\ t-1)}$ $\quad \mathbf{x} = \underline{\left(\frac{1}{t-1},\ 1\right)}$

If $t \neq 1$: $\mathcal{F}$ has 2 solutions in $(\mathbb{C}^*)^2$ - BKK bound is achieved ⚠ Even if $t=2$ (multiplicity)

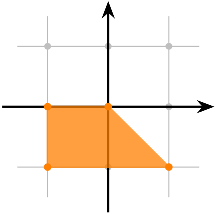
If $t = 1$: $\mathcal{F}$ has 0 solutions in $(\mathbb{C}^*)^2$ - BKK bound is not achieved

Example

$f_1(\mathbf{x}) = t + x_1 - x_2,$



$f_2(\mathbf{x}) = \frac{1}{x_1\,x_2} - \frac{1}{x_1} - \frac{x_1}{x_2} + t$



$\mathrm{MV}(P_1, P_2) = \underline{\ 2\ }$, provided $t \neq 0$. Solutions to $\mathcal{F} = \{f_1, f_2\}$:

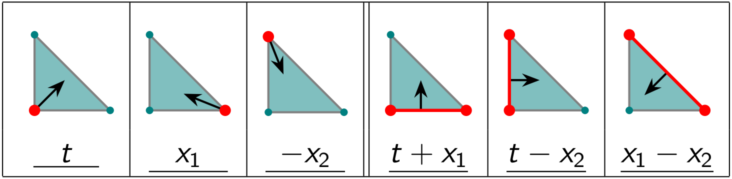
$\lvert t\rvert \neq 1$	$\mathbf{x}_- = \left(\frac{-1-t-\sqrt{5+2t+t^2}}{2}, \frac{-1+t-\sqrt{5+2t+t^2}}{2}\right)$ $\mathbf{x}_+ = \left(\frac{-1-t+\sqrt{5+2t+t^2}}{2}, \frac{-1+t+\sqrt{5+2t+t^2}}{2}\right)$	BKK is achieved (2 sol)
$t = -1$	$\mathbf{x}_- = (-1, -2),\ \mathbf{x}_+ = (1, 0) \notin (\mathbb{C}^*)^2$ (1 sol)	BKK not achieved
$t = 1$	$\{\mathbf{x} \in (\mathbb{C}^*)^2 : x_2 = 1 + x_1\}$ (0 sol)	BKK not achieved

BKK not achieved (nongeneric) $\Longleftrightarrow$ some solution outside $(\mathbb{C}^*)^\ell$ **or** continuous family of solutions

Sufficient conditions to achieve BKK bound

- $\mathcal{F} = \{f_j\}_{j=1,\dots,\ell}, \quad P_j = \mathrm{Newt}(f_j), \quad f_j(\mathbf{x}) = \sum_{\mathbf{a}\in\mathbb{Z}^\ell\cap P_j} c_{j,\mathbf{a}}\,\mathbf{x}^{\mathbf{a}}$
- Each $\mathbf{v} \in \mathbb{R}^\ell \setminus \{\mathbf{0}\}$ determines a face $F_{\mathbf{v}}(P_j)$ of P_j
  Facial system: $\mathcal{F}_{\mathbf{v}} = \{f_{j,\mathbf{v}}\}_{j=1,\dots,\ell}, \quad f_{j,\mathbf{v}}(\mathbf{x}) = \sum_{\mathbf{a}\in\mathbb{Z}^\ell\cap F_{\mathbf{v}}(P_j)} c_{j,\mathbf{a}}\,\mathbf{x}^{\mathbf{a}}$

Example: $f(\mathbf{x}) = t + x_1 - x_2$



Bernstein's Other Theorem (1975)

Suppose that no facial system $\mathcal{F}_{\mathbf{v}}$ has solutions in $(\mathbb{C}^*)^\ell$. Then all solutions to $\mathcal{F}$ are isolated and there are exactly $\mathrm{MV}(P_1, \dots, P_\ell)$ solutions to $\mathcal{F}$.

$\mathcal{F}$ has solutions outside $(\mathbb{C}^*)^\ell$ or continuous family of solutions $\iff$ BKK bound for $\mathcal{F}$ not achieved (nongeneric) $\implies$ Some $\mathcal{F}_{\mathbf{v}}$ has solutions in $(\mathbb{C}^*)^\ell$

Proof structure

Apply Bernstein's "Theorem" and "Other Theorem" to $\mathcal{F} = \{f_i\}$,

$$f_i(\mathbf{x}) = -1 + \underbrace{\frac{b_i}{2x_i} - \frac{1}{4d_i}\sum_{j,k=1}^{\ell} L_{ijk}\,\frac{2\,x_k^2 - x_i^2}{x_i\,x_j\,x_k}}_{r_i(\mathbf{x})}, \quad i = 1, \ldots, \ell.$$

For nonzero $\mathbf{b}$, $\mathbf{d}$, L,

$$P_i = \mathrm{Newt}\,(f_i) = \mathrm{conv}\big(\mathbf{0},\, \mathbf{e}_i - 2\mathbf{e}_j,\, \mathbf{e}_j - \mathbf{e}_i - \mathbf{e}_k \,:\, j, k = 1, \ldots, \ell\big).$$

Main steps

- Compute $\mathrm{MV}(P_1, \ldots, P_\ell) = D_{\ell-1}$
- Prove generic $\mathbf{b}$, $\mathbf{d}$, L are BKK generic, despite $L_{ijk} = L_{jki} = \ldots$
- Some $\mathcal{F}_{\mathbf{v}}$ has solutions in $(\mathbb{C}^*)^\ell \implies E_A(\mathrm{scal}) = 0$.

Historical note

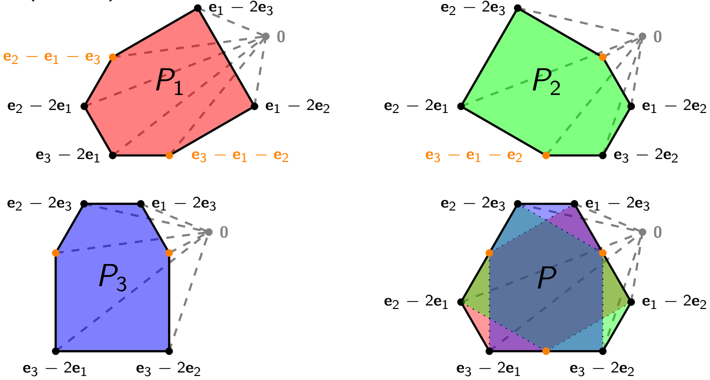
M. Graev also applied Bernstein's Theorems to Einstein equations.

$$P_i = \mathrm{Newt}\,(f_i) = \mathrm{conv}\left(\mathbf{0},\, \mathbf{e}_i - 2\mathbf{e}_j,\, \mathbf{e}_j - \mathbf{e}_i - \mathbf{e}_k \,:\; j,k = 1,\dots,\ell\right)$$

Theorem

$$\mathrm{MV}(P_1,\dots,P_\ell) = \ell!\,\mathrm{Vol}(P),\;\; P \;=\; P_1 \cup \dots \cup P_\ell$$
$$=\; \mathrm{conv}\left(\mathbf{0},\, \mathbf{e}_i - 2\mathbf{e}_j \,:\, i,j = 1,\dots,\ell\right)$$

Example ($\ell = 3$):



P is pyramid over a *permutohedron* P' $\Longrightarrow$ $\mathrm{Vol}\,P = \left(\begin{array}{c}\text{combinatorial formula}\\ \text{[Postnikov, 2009]}\end{array}\right) = \;\dots$

Theorem

Homogeneous Einstein equations are critical equations of maximum likelihood estimation problem on a scaled toric variety.

- Key facial system is $\{r_i\}_{i=1,\dots,\ell}$, $\quad r_i(\mathbf{x}) = -\frac{1}{d_i}\, x_i\, \frac{\partial\, \mathrm{scal}}{\partial x_i}$
- $$\mathrm{scal}(\mathbf{x}) = \sum_{i=1}^{\ell} d_i\, r_i(\mathbf{x}) = \sum_{i=1}^{\ell} \frac{d_i b_i}{2x_i} - \frac{1}{4} \sum_{i,j,k=1}^{\ell} L_{ijk} \frac{x_k}{x_i x_j}$$

  $A \coloneqq \mathrm{supp}(\mathrm{scal}) = (\mathbf{e}_i - 2\mathbf{e}_j)_{i,j=1,\dots,\ell}$, $\quad \mathrm{Newt}\,(\mathrm{scal}) = \mathrm{conv}(A) = P'$
- Principal A-determinant of scal is the A-resultant:
  $$E_A(\mathrm{scal}) = \mathrm{Res}_A \left(x_1 \frac{\partial\, \mathrm{scal}}{\partial x_1}, \dots, x_\ell \frac{\partial\, \mathrm{scal}}{\partial x_\ell} \right) = \prod_{F \text{ face of } P'} (\Delta_{F \cap A})^{\alpha_F}$$
- For given $\mathbf{b}$, $\mathbf{d}$, L, as $d_i > 0$,

$\exists\, \mathbf{v} \in \mathbb{R}^\ell \setminus \{\mathbf{0}\}$, $\mathcal{F}_{\mathbf{v}}$ has solutions in $(\mathbb{C}^*)^\ell$	$\Rightarrow$	$\exists\, F$ face of P', $\Delta_{F\cap A} = 0$	$\Rightarrow$	$E_A(\mathrm{scal}) = 0$

Thank you.

If $\ell = 2$:

$$r_1(\mathbf{x}) = \frac{b_1}{2x_1} - \frac{1}{4d_1}\left(\frac{L_{111}}{x_1} + \frac{2L_{112}x_2}{x_1^2} + \frac{2L_{122}}{x_1} - \frac{L_{122}x_1}{x_2^2}\right)$$
$$r_2(\mathbf{x}) = \frac{b_2}{2x_2} - \frac{1}{4d_2}\left(\frac{L_{222}}{x_2} + \frac{2L_{122}x_1}{x_2^2} + \frac{2L_{112}}{x_2} - \frac{L_{112}x_2}{x_1^2}\right)$$

$$\mathrm{scal}(\mathbf{x}) = d_1\, r_1(\mathbf{x}) + d_2\, r_2(\mathbf{x})$$

$$E_A(\mathrm{scal}) = \begin{vmatrix} L_{122} & L'_{222} & L'_{111} & L_{112} & & \\ & L_{122} & L'_{222} & L'_{111} & L_{112} & \\ & & L_{122} & L'_{222} & L'_{111} & L_{112} \\ 3L_{122} & 2L'_{222} & L'_{111} & & & \\ & 3L_{122} & 2L'_{222} & L'_{111} & & \\ & & 3L_{122} & 2L'_{222} & L'_{111} & \end{vmatrix}$$

where

$$L'_{111} = L_{111} + 2L_{122} - 2b_1 d_1$$
$$L'_{222} = L_{222} + 2L_{112} - 2b_2 d_2$$

Q: For $\ell = 2$, finiteness holds if $d_i > 0$. Is the same true for $\ell \geq 3$?

Numeric solutions on full flag manifolds G/T

G	SU(3)	SU(4)	SU(5)	SU(6)	SO(5)	SO(7)	Sp(3)	SO(8)
ℓ	3	6	10	15	4	9	9	12
$D_{\ell-1}$	13	1 683	1 462 563	7.9×10^9	63	256 729	256 729	45 046 719
BKK Bound	4	80	9 168	6 603 008	12	5 376	5 232	239 744
# solutions in $(\mathbb{C}^*)^\ell$	4	59	7 908	5 037 448	10	4 224	4 512	150 256
# solutions in $(\mathbb{R}^*)^\ell$	4	29	1 596	191 252	6	750	728	11 128
# solutions in $\mathbb{R}^\ell_+$, i.e., # G-invariant Einstein metrics on G/H	4	29	396	6572	6	48	64	184
# isometry classes of G-invariant Einstein metrics on G/H	2	4	12	35	2	5	4	5

using `HomotopyContinuation.jl`

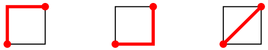
⚠ Except for $G = SU(3)$, the BKK bound is never achieved, thus, these systems are *not* generic.

Q: Find more examples of G/H where BKK bound is achieved.

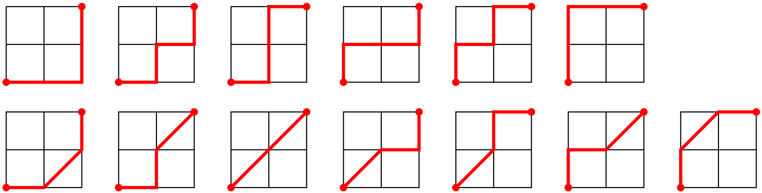
Central Delannoy number $D_m = \sum_{k=0}^{m} 2^k \binom{m}{k}^2$

Counts how many paths join opposite vertices of $m \times m$ grid, using only "right" $\rightarrow$, "up" $\uparrow$, "diagonal" $\nearrow$; e.g.,

$m = 1$:



$m = 2$:



$D_1 = 3,\quad D_2 = 13,\quad D_3 = 63,\quad D_4 = 321,$

$D_5 = 1\,683,\quad D_6 = 8\,989,\quad D_7 = 48\,639,\quad D_8 = 265\,729,\quad \ldots$