BOCHNER TECHNIQUE, REPRESENTATION THEORY, AND TWISTED SPINORS

PASADENA, CA
5/2026

§1. REPRESENTATION THEORY VIEWPOINT ON BOCHNER TECHNIQUE

Let $G \subset SO(n)$ be a Lie group with Lie algebra $\mathfrak{g} \subset \mathfrak{so}(n) \cong \Lambda^2 \mathbb{R}^n$.

$$\underset{\pi : G \longrightarrow \mathrm{Aut}(V)}{G\text{-representation}} \quad \xrightsquigarrow{\text{linearization}} \quad \underset{\pi_* : \mathfrak{g} \longrightarrow \mathrm{End}(V)}{\mathfrak{g}\text{-representation}}$$

$$V = (V_0 \oplus \cdots \oplus V_0) \oplus \underbrace{(V_1 \oplus \cdots \oplus V_1)}_{\text{"isotypic component"}} \oplus \cdots \oplus (V_k \oplus \cdots \oplus V_k), \quad V_i \ncong V_j \text{ if } i \neq j$$

repeated irreducible summands

direct sum is orthogonal

For each irreducible $\mathfrak{g}$-representation $\pi_* : \mathfrak{g} \to \mathrm{End}(V)$:

- Highest weight: $\lambda \in (\mathfrak{t}_{\mathbb{C}})^*$ uniquely identifies $\pi_* = \pi_\lambda$ up to isomorphism

  complexified Lie algebra of max. torus in G.

- Casimir eigenvalue: $\mathrm{Cas}_{\pi_*} : V \longrightarrow V$ given by $\mathrm{Cas}_{\pi_*} = -\sum_{i=1}^{\dim \mathfrak{g}} \pi_*(\alpha_i)^2$

  where (α_i) is an o.n.b. of $\mathfrak{g}$ is an equivariant endomorphism and satisfies

  $\mathrm{Cas}_{\pi_*} = \langle \lambda, \lambda + 2\rho_{\mathfrak{g}} \rangle \, \mathrm{id}_V$ where $\rho_{\mathfrak{g}} \in (\mathfrak{t}_{\mathbb{C}})^*$ is the half-sum of positive roots

Example (Type D_m) $\mathfrak{g} = \mathfrak{so}(n)$, $n = 2m$, $(\varepsilon_i)_{i=1,\dots,m}$ o.n.b. of $(\mathfrak{t}_{\mathbb{C}})^*$

Highest weights: $\lambda = a_1 \varepsilon_1 + \cdots + a_m \varepsilon_m$, where $a_1 \geq a_2 \geq \cdots \geq a_{m-1} \geq |a_m| \geq 0$

$\underbrace{a_j \in \mathbb{Z}, \forall j}_{}$ OR $\underbrace{a_j + \tfrac{1}{2} \in \mathbb{Z}, \forall j}_{}$

linearizations of irred. SO(n)-represent.

linearizations of irred. Spin(n)-representations that do not descend to SO(n)-representations

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r] & \mathrm{Spin}(n) \arrow[d] \\ (n \geq 3) & SO(n) \end{tikzcd}$$

Half-sum of positive roots: $\rho_{\mathfrak{g}} = \sum_{i=1}^{m} (m-i)\, \varepsilon_i$

λ	V	$\langle \lambda, \lambda + 2\rho_{\mathfrak{g}} \rangle$
ε_1	$\mathbb{R}^n$	$n-1$
$p \cdot \varepsilon_1$	$\mathrm{Sym}^p_0 \mathbb{R}^n$	$p(p+n-2)$
$\varepsilon_1 + \cdots + \varepsilon_p$ $p \leq m-1$	$\Lambda^p \mathbb{R}^n$	$p(n-p)$
$\frac{1}{2}\varepsilon_1 + \cdots + \frac{1}{2}\varepsilon_m$	$S^+ \mathbb{R}^n$	$\frac{n(n-1)}{8}$

Rmk: Formulas in table are also true for $n = 2m+1$ (Type B_m)

Keep!

(M^n, g) oriented Riem. mfld, curvature operator $R : \wedge^2 TM \to \wedge^2 TM$

Oriented frame bundle : $SO(n) \longrightarrow F$

(principal $SO(n)$–bundle)

$$\begin{tikzcd} SO(n) \arrow[r] & F \arrow[d] \\ & M \end{tikzcd}$$

Orthogonal representation :

$$\pi : SO(n) \longrightarrow Aut(V)$$

$\rightsquigarrow$ Associated bundle

$$\begin{tikzcd} V \arrow[r] & VM = F \times_\pi V \arrow[d] \\ & M \end{tikzcd}$$

On sections of VM, we define a 'Laplacian' as

$$\Delta = \nabla^* \nabla + t\, K(R, \pi_*)$$

where $\nabla^*\nabla$ is the connection Laplacian, $t \in \mathbb{R}$ and

$$K(R, \pi_*) = - \sum_{i=1}^{\binom{n}{2}} \pi_*(R(\alpha_i)) \circ \pi_*(\alpha_i) \in End(VM)$$

Petersen writes

$Ric(T) = K(R, \pi_*)T$

where $\pi_* : so(n) \to End((\mathbb{R}^n)^{\otimes p})$

$$= - \sum_{i,j=1}^{\binom{n}{2}} R_{ij}\, \pi_*(\alpha_i) \circ \pi_*(\alpha_j), \text{ if } R = \sum_{i,j=1}^{\binom{n}{2}} R_{ij}\, \alpha_i \otimes \alpha_j \in End(so(n))$$

where $(\alpha_i)_{i=1,\dots,\binom{n}{2}}$ is an orthonormal basis of $so(n) \cong \wedge^2 \mathbb{R}^n$

e.g., lexicographic basis $e_i \wedge e_j$, $i<j$

w.r.t. $Q(X,Y) = \frac{1}{2} tr\, XY^T$

via: $\forall v, w \in \mathbb{R}^n$ $(v \wedge w)(\cdot) = \langle v, \cdot \rangle w - \langle w, \cdot \rangle v$

More generally, if structure gp. of frame bundle reduces to $G \subset SO(n)$,

e.g., if $G = Hol(M^n, g)$

$$\begin{tikzcd} G \arrow[r] & F' \arrow[d] \\ & M \end{tikzcd}, \quad \begin{array}{l} \pi : G \longrightarrow Aut(V) \\ \pi_* : \mathfrak{g} \longrightarrow End(V) \end{array}, \quad R = \begin{bmatrix} R_\mathfrak{g} & 0 \\ 0 & 0 \end{bmatrix} : \mathfrak{g} \oplus \mathfrak{g}^\perp \longrightarrow \mathfrak{g} \oplus \mathfrak{g}^\perp$$

$(R_\mathfrak{g} : \mathfrak{g} \longrightarrow \mathfrak{g})$

and we can use an orthonormal basis $(\alpha_i)_{i=1,\dots,\dim \mathfrak{g}}$ to define

$$K_\mathfrak{g}(R, \pi_*) = - \sum_{i=1}^{\dim \mathfrak{g}} \pi_*(R_\mathfrak{g}(\alpha_i)) \circ \pi_*(\alpha_i)$$

(If $\mathfrak{g} = so(n)$, write $K(R, \pi_*)$!)

E.g., if (M^n, g) is Kähler, then $n = 2m$, $G = U(m)$, $R_\mathfrak{g}$ "Kähler curv. op."

Properties of $K_{\mathfrak{g}}(R,\pi_*)$:

$$\langle K_{\mathfrak{g}}(R,\pi_*)v, v\rangle = -\sum_i \langle \pi_*(\underset{\nu_i\alpha_i}{\underset{\shortparallel}{R\alpha_i}})\pi_*(\alpha_i)v, v\rangle = \sum_i \nu_i \cdot \|\pi_*(\alpha_i)v\|^2 \geq 0, \forall v$$

(1) $K_{\mathfrak{g}}(R,\pi_*) \in \mathrm{End}(V)$ is self-adjoint

(2) $R \geq 0 \Rightarrow K_{\mathfrak{g}}(R,\pi_*) \geq 0$ for all orthogonal $\mathfrak{g}$-representations π_*

(3) $\mathrm{End}(\mathfrak{g}) \ni R \longmapsto K_{\mathfrak{g}}(R,\pi_*) \in \mathrm{End}(V)$ is G-equivariant

$\mathrm{Ad}_G \circlearrowleft$ $\qquad$ $\pi \circlearrowleft$

(4) If $\pi^*: \mathfrak{g} \to \mathrm{End}(V^*)$ is the 'dual' of $\pi: \mathfrak{g} \to \mathrm{End}(V)$, then $K_{\mathfrak{g}}(R,\pi^*) = \left(K_{\mathfrak{g}}(R,\pi)\right)^*$

(5) If $\pi_* = \pi_1 \oplus \pi_2$, then $K_{\mathfrak{g}}(R,\pi_*) = \mathrm{diag}\left(K_{\mathfrak{g}}(R,\pi_1), K_{\mathfrak{g}}(R,\pi_2)\right)$ 'block-diagonal' endomorphism on $V = V_1 \oplus V_2$

(6) If $\pi_*: \mathfrak{g} \to \mathrm{End}(V)$ is irreducible with highest weight λ, then

$$K_{\mathfrak{g}}(\mathrm{id}_{\mathfrak{g}}, \pi_*) = \langle \lambda, \lambda + 2\rho_{\mathfrak{g}} \rangle \, \mathrm{id}_V$$

General Bochner technique: If (M^n,g) is closed and $\Delta\alpha = 0$, then

$$0 = \int_M \langle \Delta\alpha, \alpha\rangle = \int_M \|\nabla\alpha\|^2 + t\langle K(R,\pi_*)\alpha,\alpha\rangle$$

Let $VM = V_0M \oplus V_1M \oplus \cdots \oplus V_kM$ be the isotypic components of π_*,

$\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_k$

where V_0M corresponds to the trivial component. Then

$$K(R,\pi_*) = \mathrm{diag}\left(0, K_{\mathfrak{g}}(R,\pi_{\lambda_1}), \ldots, K_{\mathfrak{g}}(R,\pi_{\lambda_k})\right)$$

Note: $\mathrm{Ker}\, K(R,\pi_*) \neq \{0\}$ if π_* has trivial summands.

so $t\,K(R,\pi_*) \geq 0 \iff t\,K_{\mathfrak{g}}(R,\pi_{\lambda_j}) \geq 0 \ \forall j = 1,\ldots,k \Rightarrow \begin{cases} \nabla\alpha_j \equiv 0 & \forall j = 0,1,\ldots,k \\ \alpha_j \in \mathrm{Ker}\, t\,K(R,\pi_{\lambda_j}) & \forall j = 1,\ldots,k \end{cases}$

$t \cdot K(R,\pi_{\lambda_j}) > 0 \ \forall j = 1,\ldots,k \Rightarrow \alpha_j = 0 \ \forall j = 1,\ldots,k \Rightarrow$ $\alpha = \alpha_0 \in V_0M$ takes values in the subbundle corresponding to trivial isotypic component. (so $\alpha = 0$ if $V_0M = \{0\}$).

Classical examples: Bochner 1946: $\mathrm{Ric} > 0 \Rightarrow b_1(M,\mathbb{R}) = 0$

(also have versions for ≥ 0 instead of > 0)

Meyer, Gallot, 1970's: $R > 0 \Rightarrow b_p(M,\mathbb{R}) = 0, \forall p \leq \frac{n}{2}$ $\Rightarrow$ $M \cong_{\mathbb{Q}} S^n$ is a rational homology sph

Tachibana 1974: $R > 0$ and $\underbrace{\mathrm{div}\, R = 0}_{\Updownarrow}$ $\Rightarrow (M^n,g)$ has $\sec \equiv K$.

$\Delta R = 0.$ $\leftarrow$ b/c $d^\nabla R = (dR + \theta \cdot R)_{\text{skew}} = 0$, $\Delta = (d^\nabla + \mathrm{div})^2$

Q: When is $K_{\mathfrak{g}}(R,\pi_*) \geqslant 0$? Sufficient (and necessary?) conditions?

↖ e.g., refining the above?

Hitchin 2015: $R \geqslant 0 \iff K(R,\pi_*) \geqslant 0$, $\forall$ irreducible $\mathfrak{so}(n)$-rep. π_*

B.–Mendes 2022: $\sec \geqslant 0 \iff K(R,\pi_{p\varepsilon_1}) \geqslant 0$, $\forall p \geqslant 2$

↑ recall: this is $\mathrm{Sym}^p_0 \mathbb{R}^n$

$K(R,\pi_{\varepsilon_1}) = \mathrm{Ric}_R$: $\mathrm{Ric} \geqslant 0 \iff K(R,\pi_{\varepsilon_1}) \geqslant 0$

$K(R,\pi_S) = \dfrac{\mathrm{scal}_R}{8}$: $\mathrm{scal} \geqslant 0 \iff K(R,\pi_S) \geqslant 0$

Question 1: Given a curvature condition ($O(n)$-invariant cone in $\mathrm{Sym}^2_b \Lambda^2 \mathbb{R}^n$) find a collection of $\mathfrak{so}(n)$-representations that 'characterize' it via $K(R,\pi_*)$.

(I have some initial thoughts about PIC and $\sec_{\mathbb{C}} \geqslant 0$.)

Conjecture: $K(R,\pi_{p\varepsilon_1}) \geqslant 0 \implies K(R,\pi_{(p-1)\varepsilon_1}) \geqslant 0$ $\forall p \geqslant 1$.

Note: Open for $p \geqslant 3$. Case $p=2$ is due to Berger, case $p=1$ is trivial.

§2. SUFFICIENT CONDITIONS VIA r-POSITIVE CURVATURE OPERATOR

Given $R: \mathfrak{g} \to \mathfrak{g}$ alg. curvature operator, order its eigenvalues:

$$\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{\dim \mathfrak{g}}$$

set $\Sigma(r,R) := \nu_1 + \cdots + \nu_{\lfloor r \rfloor} + (r - \lfloor r \rfloor)\nu_{\lfloor r \rfloor + 1}$; say R is r-positive if $\Sigma(r,R) > 0$.

Note: $\Sigma(1,R) > 0 \iff R > 0$, $\Sigma(\dim \mathfrak{g}, R) > 0 \iff \mathrm{tr}\, R > 0$

this is $\Lambda^p \mathbb{R}^n$, $1 \leq p \leq \lfloor n/2 \rfloor$

Thm (Petersen–Wink, 2021). $\Sigma(n-p, R) > 0 \implies K(R, \pi_{\varepsilon_1 + \cdots + \varepsilon_p}) > 0$

As a consequence, if (M^n, g) has $(n-p)$-positive curvature operator, then

$$b_1(M) = \cdots = b_p(M) = 0 \quad \text{and} \quad b_{n-p}(M) = \cdots = b_n(M) = 0$$

Def. The PW-invariant of an irreducible $\mathfrak{g}$-representation $\pi_\lambda : \mathfrak{g} \to \mathrm{End}(V)$ with highest weight λ is:

Question: is this ever the minimum?

$$PW_{\mathfrak{g}}(\pi_\lambda) = \min \left\{ \frac{\langle \lambda, \lambda + 2\rho_{\mathfrak{g}} \rangle}{\langle \lambda, \lambda \rangle}, \ \dim \mathfrak{g} \right\}$$

The PW-invariant of a $\mathfrak{g}$-representation $\pi_* : \mathfrak{g} \to \mathrm{End}(V)$ is:

$$PW_{\mathfrak{g}}(\pi) = \min \left\{ PW_{\mathfrak{g}}(\pi_\lambda) : \pi_\lambda \text{ nontrivial irreducible summand in } \pi_* \right\}$$

Proposition (B.–Goodman, 2024). Given $R : \mathfrak{g} \to \mathfrak{g}$ self-adjoint and $\pi_\lambda : \mathfrak{g} \to \mathrm{End}(V)$,

$$K_{\mathfrak{g}}(R, \pi_\lambda) \geq \|\lambda\|^2 \, \Sigma\left(PW_{\mathfrak{g}}(\pi_\lambda), R\right) \cdot \mathrm{id}_V$$

Cor: If R is $PW_{\mathfrak{g}}(\pi_*)$-positive, then $K_{\mathfrak{g}}(R, \pi_*) > 0$.

Pf: Let (α_i) be o.n.b. of eigenvectors of R, where $R\alpha_i = \nu_i \alpha_i$ and $\nu_1 \leq \dots \leq \nu_{\dim \mathfrak{g}}$. Suppose $PW_{\mathfrak{g}}(\pi_\lambda) < \dim \mathfrak{g}$, let $r = \lfloor PW_{\mathfrak{g}}(\pi_\lambda) \rfloor$. Then:

$$\langle K_{\mathfrak{g}}(R, \pi_*) v, v \rangle = \sum_{i=1}^{\dim \mathfrak{g}} \langle \pi_* R(\alpha_i) v, \pi_*(\alpha_i) v \rangle$$

$$= \sum_{i=1}^{\dim \mathfrak{g}} \nu_i \|\pi_*(\alpha_i) v\|^2$$

split into last r terms and use $\nu_i \leq \nu_j$ if $i \leq j$

$$\geq \sum_{i=1}^{r} \nu_i \|\pi_*(\alpha_i) v\|^2 + \underbrace{\sum_{i=r+1}^{\dim \mathfrak{g}} \nu_{r+1} \|\pi_*(\alpha_i) v\|^2}_{\nu_{r+1} \cdot \left(\|\mathrm{cas}_{\pi_*}(v)\|^2 - \sum_{i=1}^{r} \|\pi_*(\alpha_i) v\|^2 \right)}$$

add back first r terms in second sum

$$= -\sum_{i=1}^{r} \underbrace{(\nu_{r+1} - \nu_i)}_{\geq 0} \|\pi_*(\alpha_i) v\|^2 + \nu_{r+1} \langle \lambda, \lambda + 2\rho_{\mathfrak{g}} \rangle \|v\|^2$$

$\|\pi_*(\alpha_i) v\| \leq \|\lambda\| \|v\|$ b/c, up to conjugating, may assume $\alpha_i \in \mathfrak{t}^*_{\mathbb{C}}$ is in Cartan subalgebra, whose eigenvalues of $\pi_*(\alpha_i)$ are $\mu(\alpha_i)$ for each weight $\mu \in \mathfrak{t}^*_{\mathbb{C}}$, and $|\mu(\alpha_i)| \leq \|\mu\| \cdot \underbrace{\|\alpha_i\|}_{=1} \leq \|\lambda\|$ ↑ λ is highest weight

$$\geq -\sum_{i=1}^{r} \underbrace{(\nu_{r+1} - \nu_i)}_{\geq 0} \|\lambda\|^2 \|v\|^2 + \nu_{r+1} \langle \lambda, \lambda + 2\rho_{\mathfrak{g}} \rangle \|v\|^2$$

$$= \|\lambda\|^2 \underbrace{\left(\sum_{i=1}^{r} \nu_i + \left(\mathrm{PW}_{\mathfrak{g}}(\pi_\lambda) - r \right) \nu_{r+1} \right)}_{\Sigma(\mathrm{PW}_{\mathfrak{g}}(\pi_\lambda),\, R)} \|v\|^2 .$$

If, instead, $\mathrm{PW}_{\mathfrak{g}}(\pi_\lambda) = \dim \mathfrak{g} \leq \dfrac{\langle \lambda, \lambda + 2\rho_{\mathfrak{g}} \rangle}{\langle \lambda, \lambda \rangle}$, then:

$\nu_{\dim \mathfrak{g}} \geq 0$: similar argument using $\nu_{\dim \mathfrak{g}}$ instead of ν_{r+1} in second sum,

$\nu_{\dim \mathfrak{g}} < 0$: use $\| \pi_*(\alpha_i) v \| \leq \|\lambda\| \cdot \|v\|$. ☐

§3. APPLICATIONS TO VANISHING THEOREMS

λ	V	$\langle \lambda, \lambda + 2\rho_{\mathfrak{g}} \rangle$	$\mathrm{PW}_{\mathfrak{so}(n)}(\pi_\lambda)$
ε_1	$\mathbb{R}^n$	$n-1$	$n-1$
$p \cdot \varepsilon_1$	$\mathrm{Sym}^p_0 \mathbb{R}^n$	$p(p+n-2)$	$\frac{p+n-2}{p}$
$\varepsilon_1 + \dots + \varepsilon_p$ $p \leq m-1$	$\Lambda^p \mathbb{R}^n$	$p(n-p)$	$n-p$
$\frac{1}{2}\varepsilon_1 + \dots + \frac{1}{2}\varepsilon_m$	$S^+ \mathbb{R}^n$	$\frac{n(n-1)}{8}$	$n-1$

(recovers result of Petersen-Wink, 2021)

Note: R $(n-1)$-pos $\Rightarrow \mathrm{Ric} > 0$

Tachibana-type result [Petersen-Wink, 2021]: $V = \mathrm{Sym}^2_b(\Lambda^2 \mathbb{R}^n)$ algebraic curvature operators

$$\pi \cong \begin{cases} \overset{\mathrm{scal}}{\pi_0} \oplus \overset{\mathring{\mathrm{Ric}}}{\pi_{2\varepsilon_1}} \oplus \overset{W}{\pi_{2\varepsilon_1 + 2\varepsilon_2}}, & n \geq 5 \\ \pi_0 \oplus \pi_{2\varepsilon_1} \oplus \pi_{\varepsilon_1+\varepsilon_2} \oplus \pi_{\varepsilon_1 - \varepsilon_2}, & n = 4 \\ \pi_0 \oplus \pi_{2\varepsilon_1}, & n = 3 \end{cases} \Rightarrow \mathrm{PW}_{\mathfrak{so}(n)}(\pi) = \begin{cases} \frac{n-1}{2}, & n \geq 5 \\ 2, & n = 4 \\ \frac{3}{2}, & n = 3 \end{cases}$$

(minimum is achieved by the red irreducible)

If (M^n, g) has harmonic curvature operator $(\Delta R = 0 \Leftrightarrow \operatorname{div} R = 0)$ and:

$\mathrm{PW}_{\mathfrak{so}(n)}(\pi)$-nonnegative $R \Rightarrow \nabla R = 0 \Leftrightarrow (M^n, g)$ is locally symmetric

$\mathrm{PW}_{\mathfrak{so}(n)}(\pi)$ - positive $R \Rightarrow R = R_0 \in V_0 M = \{ K \cdot \mathrm{Id} \} \Leftrightarrow (M^n, g)$ has $\sec \equiv K$.

Question 2: Are there similar results for $\Delta(\nabla R) = 0$ using $V \subset \mathrm{Sym}^2_b(\Lambda^2 \mathbb{R}^n) \otimes \mathbb{R}^n$?

↑ 2nd Bianchi identity

[Petersen–Wink, 2021]: If (M^{2m}, g) is Kähler and $R_{\mathfrak{u}(m)} : \mathfrak{u}(m) \longrightarrow \mathfrak{u}(m)$ is $C^{p,q}$-positive

$$C^{p,q} = m+1 - \frac{p^2+q^2}{p+q}, \quad \text{then} \quad h^{p,q}(M) = 0.$$

Pf: $PW_{\mathfrak{u}(m)}\left(\Lambda^{p,q}\,\mathbb{C}^m\right) = C^{p,q}$, $\quad h^{p,q}(M) = \dim \operatorname{Ker}\left(\Delta : \Lambda^{p,q}TM \to \Lambda^{p,q}TM\right)$

Q: What other applications yield topological conclusions?

Non-example: $\pi_{p\cdot\varepsilon_1}$: (M^n, g), $\frac{p+n-2}{p}$-positive $R \Longrightarrow \dim \operatorname{Ker}\left(\Delta : \operatorname{Sym}^p_0 TM \to \operatorname{Sym}^p_0 TM\right) = 0.$

⚠ This dimension depends on the metric g, not just M.

'All' examples: $\pi_S \otimes \pi_*$ twisted spinors ← instead of $\dim \operatorname{Ker} \Delta = 0$, factor $\Delta = D^2$ and get $\operatorname{ind}(D) = 0$, which is topological by Atiyah-Singer!

§4. BOCHNER TECHNIQUE FOR TWISTED SPINORS

Spinor frame bundle:

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r] & F' \arrow[d] \\ & M \end{tikzcd}$$

Henceforth, assume M is spin

Spin representation: $\mu_S : \mathrm{Spin}(n) \longrightarrow \mathrm{Aut}(S)$ ← $\dim S = 2^n$

$\pi_S : \mathfrak{so}(n) \longrightarrow \mathrm{End}(S)$

Any representation of $\mathfrak{so}(n)$: $\pi_* : \mathfrak{so}(n) \longrightarrow \mathrm{End}(V)$ is the linearization of some $\mu_* : \mathrm{Spin}(n) \to \mathrm{Aut}(V)$

$\Longrightarrow$

Spinor bundle

$$\begin{tikzcd} S \arrow[r] & SM = F' \times_{\mu_S} S \arrow[d] \\ & M \end{tikzcd}$$

Twisted spinor bundle:

$$\begin{tikzcd} S\otimes V \arrow[r] & SM \otimes VM = F' \times_{\mu_S \otimes \mu_*} (S \otimes V) \arrow[d] \\ & M \end{tikzcd}$$

Dirac operator (on spinor bundle SM): $D : SM \longrightarrow SM$

$$\phi \longmapsto \sum_{j=1}^{n} e_j \cdot \nabla_{e_j} \phi$$

connection induced from Levi-Civita

Clifford multiplication

$$D^2 = \nabla^* \nabla + \frac{\mathrm{scal}}{4}, \quad \text{so } \mathrm{scal} > 0 \Longrightarrow \operatorname{Ker} D = \{0\} \xRightarrow[\text{index thm}]{AS} \underbrace{\hat{A}(M) = 0}_{\text{topological conclusion!}}.$$

$\hat{A}(M) =$ linear comb. of Pontryagin #s.

Twisted Dirac operator (on $SM \otimes VM$): $D_V : SM \otimes VM \to SM \otimes VM$

$$\phi \otimes v \longmapsto \sum_{j=1}^{n} \left(e_j \cdot \nabla_{e_j} \phi\right) \otimes v + \left(e_j \cdot \phi\right) \otimes \nabla_{e_j} v$$

$$D_V^2 = \nabla^* \nabla + \mathcal{R}_V, \quad \text{so } \mathcal{R}_V > 0 \Longrightarrow \operatorname{Ker} D_V = \{0\} \xRightarrow[\text{index thm}]{AS} \underbrace{\hat{A}(M, V_{\mathbb{C}}) = 0}_{\text{also topological:}}$$

$= \langle \hat{A}(TM) \cdot ch(VM_{\mathbb{C}}), [M] \rangle$

Prop (B.–Goodman 2024) $\mathcal{R}_V = K(R, \pi_S \otimes \pi_*) + \underbrace{K(R, \pi_S)}_{\frac{\text{scal}}{8}} - K(R, \pi_*)$

Examples:

- V trivial $\Rightarrow \mathcal{R}_V = 2K(R, \pi_S) = \frac{\text{scal}}{4}$
- $V \cong S \Rightarrow SM \otimes VM \cong \bigoplus_{p=0}^{n} \Lambda^p TM$ and $\mathcal{R}_V = \text{diag}\Big(\overbrace{K(R, \pi_{\varepsilon_1 + \dots + \varepsilon_p})}^{K(R, \pi_S \otimes \pi_S)}\Big)$

$D_S^2 = \Delta$ is the Hodge Laplacian

(above is for $n=2m$, if $n=2m+1$ then $SM \otimes VM \cong \bigoplus_{p=0}^{m} \Lambda^{2p} TM$ but 'recover' everything by Poincaré duality)

$$\begin{array}{l} \text{diag}\big(\underset{\uparrow \atop \Lambda^0 TM}{0}, \underset{\uparrow \atop \Lambda^1 TM}{K(R, \pi_{\varepsilon_1})}, \underset{\uparrow \atop \Lambda^2 TM \dots}{K(R, \pi_{\varepsilon_1+\varepsilon_2})}, \dots, K(R, \pi_{\varepsilon_1+\dots+\varepsilon_m}), \\ \quad \underset{\uparrow \atop \dots \Lambda^{m+1} TM}{K(R, \pi_{\varepsilon_1+\dots+\varepsilon_m})}, \dots, \underset{\uparrow \atop \dots \Lambda^{n-1} TM}{K(R, \pi_{\varepsilon_1})}, \underset{\uparrow \atop \Lambda^n TM}{0}\big) \end{array}$$

But, in general, $\mathcal{R}_V > 0$ is a 'pinching' condition on R!

Using PW–type estimate on $K(R, \cdot)$, we find $C_p(R)$ for each $p \geq 1$ s.t.

piecewise linear comb. of eigenvalues of R

Thm (B.–Goodman, 2024). If (M^n, g) has $C_p(R) > 0$ and $V \subseteq TM^{\otimes p}$ is a parallel subbundle, then $\hat{A}(M, V_{\mathbb{C}}) = 0$.

some linear comb. of Pontryagin numbers

Similar results give conditions on R that imply M $\begin{cases} \text{is null-cobordant} \\ \text{has vanishing Witten genus} \\ \text{has vanishing signature} \end{cases}$ …

Question 3: Similar results in special holonomy?

(Kähler or quaternion Kähler)

§5. BOCHNER TECHNIQUE IN COMPARISON GEOMETRY

Thm (Llarull, 1998). The round sphere (S^n, g_1) is 'area-extremal' for scal: if g has $\text{scal}_g \geq \text{scal}_{g_1}$ and $\Lambda^2 g \geq \Lambda^2 g_1$, then $\text{scal}_g = \text{scal}_{g_0}$. Actually, $(S^n, g) \underset{\text{isom.}}{\cong} (S^n, g_1)$.

Pf: Build 'mixed' Dirac operator D_{g, g_1} and use $\Lambda^2 g \geq \Lambda^2 g_1$ to show:

$D_{g,g_1}^2 \geq D^* D + \frac{1}{4}(\text{scal}_g - \text{scal}_{g_1})$. Index theory gives $0 \neq \phi \in \text{Ker}\, D_{g, g_1}$

so $0 = \int_M (D_{g,g_1}^2 \phi, \phi) \geq \int_M \|D\phi\|^2 + \frac{1}{4}\underbrace{(\text{scal}_g - \text{scal}_{g_0})}_{\geq 0}\|\phi\|^2 \Rightarrow \text{scal}_g = \text{scal}_{g_0}$. ☐

Goette–Semmelmann 2002: Area-extremality holds if $R \geq 0$ and $\chi(M) \neq 0$.

If $n=4$, then sec>0 & … is enough. [B.–Goodman, 23]

Question 4 (Goodman): Does area-extremality hold if R is r-positive?