Counting homogeneous Einstein metrics

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Joint work with Hannah Friedman

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Homogeneous Einstein metrics

- G/H compact homogeneous space
- Fix bi-invariant metric Q on G and $\mathfrak{m} := \mathfrak{h}^\perp \subset \mathfrak{g}$
- Decompose into H-irreducibles $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\ell$
- Assume $\mathfrak{m}_i \ncong \mathfrak{m}_j$, for all $i \neq j$
- Every G-invariant Riemannian metric g is of the form

$$\mathrm{g} = x_1\, Q|_{\mathfrak{m}_1} + \cdots + x_\ell\, Q|_{\mathfrak{m}_\ell},$$

for some $\mathbf{x} = (x_1, \ldots, x_\ell)$, $x_i > 0$, and has Ricci tensor

$$\begin{aligned}\mathrm{Ric}_{\mathrm{g}} &= r_1(\mathbf{x})\, x_1\, Q|_{\mathfrak{m}_1} + \cdots + r_\ell(\mathbf{x})\, x_\ell\, Q|_{\mathfrak{m}_\ell} \\ &= r_1(\mathbf{x})\, \mathrm{g}|_{\mathfrak{m}_1} + \cdots + r_\ell(\mathbf{x})\, \mathrm{g}|_{\mathfrak{m}_\ell}\end{aligned}$$

where

$$\boxed{r_i(\mathbf{x}) = \frac{b_i}{2x_i} - \frac{1}{4d_i} \sum_{j,k=1}^{\ell} L_{ijk} \frac{2\,x_k^2 - x_i^2}{x_i\, x_j\, x_k}}$$

$$r_i(\mathbf{x}) = \frac{b_i}{2x_i} - \frac{1}{4d_i} \sum_{j,k=1}^{\ell} L_{ijk} \frac{2\,x_k^2 - x_i^2}{x_i\, x_j\, x_k}$$

are Laurent polynomials with 'parameters' $b_i \geq 0$, $d_i > 0$, $L_{ijk} \geq 0$,

$$B|_{\mathfrak{m}_i} = -b_i\, Q|_{\mathfrak{m}_i}, \quad d_i := \dim \mathfrak{m}_i, \quad L_{ijk} := \sum_{\substack{\mathbf{v}_\alpha \in \mathfrak{m}_i,\, \mathbf{v}_\beta \in \mathfrak{m}_j \\ \mathbf{v}_\gamma \in \mathfrak{m}_k}} Q([\mathbf{v}_\alpha, \mathbf{v}_\beta], \mathbf{v}_\gamma)^2$$

$\mathbf{b} = (b_i)$, $\mathbf{d} = (d_i)$, $L = (L_{ijk})$ depend on G/H and $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\ell$

$$\mathrm{g} \;\text{ is Einstein} \iff \mathrm{Ric}_{\mathrm{g}} = \cancel{c}^{\,1}\, \mathrm{g} \iff r_i(\mathbf{x}) = \cancel{c}^{\,1},\ \forall i$$

Existence Question

Does $r_1(\mathbf{x}) = \cdots = r_\ell(\mathbf{x}) = 1$ admit solutions $\mathbf{x} \in \mathbb{R}_+^\ell$?

Finiteness Conjecture (Böhm–Wang–Ziller, 2004)

There are **finitely many** solutions to $r_1(\mathbf{x}) = \cdots = r_\ell(\mathbf{x}) = 1$.

Main results (Geometric version)

Theorem

If G/H *is a compact homogeneous space whose isotropy representation* $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\ell$ *consists of* ℓ *pairwise inequivalent irreducible summands, then there are at most*

$$D_{\ell-1} = \sum_{k=0}^{\ell-1} 2^k \binom{\ell-1}{k}^2$$

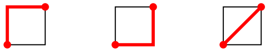
isolated G*-invariant Einstein metrics* g *on* G/H *with* $\mathrm{Ric}_{\mathrm{g}} = \mathrm{g}$.

All G*-invariant Einstein metrics with* $\mathrm{Ric}_{\mathrm{g}} = \mathrm{g}$ *are isolated if* $E_A(\mathrm{scal}) \neq 0$, *so the Finiteness Conjecture holds in such cases.*

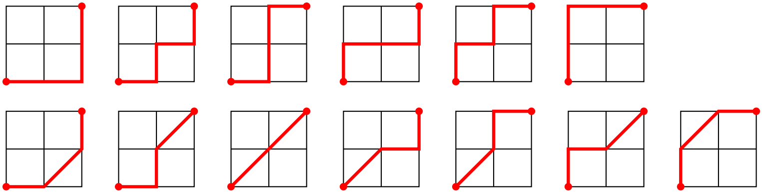
Central Delannoy number $D_m = \sum_{k=0}^{m} 2^k \binom{m}{k}^2$

Counts how many paths join opposite vertices of $m \times m$ grid, using only "right" $\rightarrow$, "up" $\uparrow$, "diagonal" $\nearrow$; e.g.,

$m = 1$:



$m = 2$:



$D_1 = 3$, $D_2 = 13$, $D_3 = 63$, $D_4 = 321$,
$D_5 = 1\,683$, $D_6 = 8\,989$, $D_7 = 48\,639$, $D_8 = 265\,729$, $\ldots$

Main results (Algebraic version)

Theorem

(i) *For all* $\mathbf{b}$, $\mathbf{d}$, *and* L, *there are at most* $D_{\ell-1}$ *isolated solutions* $\mathbf{x} \in (\mathbb{C}^*)^\ell$ *to the Einstein equations* $r_1(\mathbf{x}) = \cdots = r_\ell(\mathbf{x}) = 1$.

(ii) *For generic* $\mathbf{b}$, $\mathbf{d}$, *and* L, *all solutions to the Einstein equations are isolated and there are exactly* $D_{\ell-1}$ *solutions in* $(\mathbb{C}^*)^\ell$.

(iii) *For each support* A, *there is a polynomial* $E_A(\mathrm{scal})$ *on* $\mathbf{b}$, $\mathbf{d}$, *and* L *such that* $E_A(\mathrm{scal}) \neq 0$ *implies* $\mathbf{b}$, $\mathbf{d}$, *and* L *are generic.*

$$r_i(\mathbf{x}) = \frac{b_i}{2x_i} - \frac{1}{4d_i} \sum_{j,k=1}^{\ell} L_{ijk} \frac{2\,x_k^2 - x_i^2}{x_i\,x_j\,x_k}, \quad i = 1, \ldots, \ell$$

Ingredients in the proof

Bernstein–Khovanskii–Kushnirenko

Bernstein's Theorem (1975), "BKK bound"

The system $\mathcal{F} = \{f_1, \ldots, f_\ell\}$ of Laurent polynomials has at most $\mathrm{MV}(P_1, \ldots, P_\ell)$ isolated solutions in $(\mathbb{C}^*)^\ell$, where $P_j = \mathrm{Newt}\,(f_j)$.

Bernstein's Other Theorem (1975)

If no facial system $\mathcal{F}_{\mathbf{v}}$ has solutions in $(\mathbb{C}^*)^\ell$, then all solutions to $\mathcal{F}$ are isolated and there are exactly $\mathrm{MV}(P_1, \ldots, P_\ell)$ solutions to $\mathcal{F}$.

Main steps

- Compute $\mathrm{MV}(P_1, \ldots, P_\ell) = D_{\ell-1}$. [Using Postnikov's formula]
- Prove generic $\mathbf{b}$, $\mathbf{d}$, L are BKK generic, despite $L_{ijk} = L_{jki} = \ldots$
- Some $\mathcal{F}_{\mathbf{v}}$ has solutions in $(\mathbb{C}^*)^\ell \implies E_A(\mathrm{scal}) = 0$.

Thank you.

Support and Newton polytope

$$\mathbf{x} = (x_1, \ldots, x_\ell) \in (\mathbb{C}^*)^\ell$$
$$\mathbf{a} = (a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell$$
$$\mathbf{x}^\mathbf{a} := x_1^{a_1} \ldots x_\ell^{a_\ell}$$

Define, for a Laurent polynomial

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in \mathbb{Z}^\ell} c_\mathbf{a}\, \mathbf{x}^\mathbf{a},$$

- Support:

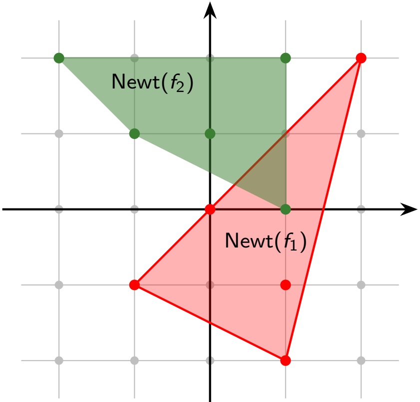
  $\operatorname{supp} f := \{\mathbf{a} \in \mathbb{Z}^\ell : c_\mathbf{a} \neq 0\}$
- Newton polytope:

  $\operatorname{Newt}(f) := \operatorname{conv}(\operatorname{supp} f)$

Example

$$f_1(\mathbf{x}) = \frac{\sqrt{5}}{x_1 x_2} + 3\frac{x_1}{x_2^2} - \frac{x_1}{x_2} + 1 - x_1^2 x_2^2$$

$$f_2(\mathbf{x}) = \frac{x_2^2}{x_1^2} + \frac{x_2}{x_1} - x_1 + 8x_2 + x_1 x_2^2$$



Mixed volume

Given $P_1, \ldots, P_\ell \subset \mathbb{R}^\ell$ polytopes, $\lambda_1, \ldots, \lambda_\ell > 0$, the volume of

$$\lambda_1 P_1 + \cdots + \lambda_\ell P_\ell := \underline{\left\{ \sum_{j=1}^{\ell} \lambda_j \mathbf{p}_j : \mathbf{p}_j \in P_j \right\}}$$

is a homogeneous polynomial $V(\lambda_1, \ldots, \lambda_\ell)$ of degree ℓ.

Definition (Mixed volume)

$\mathrm{MV}(P_1, \ldots, P_\ell)$ is the coefficient of $\lambda_1 \ldots \lambda_\ell$ in $V(\lambda_1, \ldots, \lambda_\ell)$.

- P_j lattice polytopes $\implies \mathrm{MV}(P_1, \ldots, P_\ell)$ is an integer.
- $\mathrm{MV}(P_1, \ldots, P_\ell)$ is the unique symmetric multilinear function such that $\mathrm{MV}(P, \ldots, P) = \ell!\ \mathrm{Vol}(P)$.

Theorem

Homogeneous Einstein equations are critical equations of maximum likelihood estimation problem on a scaled toric variety.

- Key facial system is $\{r_i\}_{i=1,\dots,\ell}$, $\quad r_i(\mathbf{x}) = -\frac{1}{d_i}\, x_i\, \frac{\partial\,\mathrm{scal}}{\partial x_i}$
- $$\mathrm{scal}(\mathbf{x}) \;=\; \sum_{i=1}^{\ell} d_i\, r_i(\mathbf{x}) \;=\; \sum_{i=1}^{\ell} \frac{d_i b_i}{2x_i} - \frac{1}{4} \sum_{i,j,k=1}^{\ell} L_{ijk} \frac{x_k}{x_i x_j}$$

  $A \coloneqq \mathrm{supp}(\mathrm{scal}) = (\mathbf{e}_i - 2\mathbf{e}_j)_{i,j=1,\dots,\ell}$, $\quad \mathrm{Newt}\,(\mathrm{scal}) = \mathrm{conv}(A) = P'$
- Principal A-determinant of scal is the A-resultant:

  $$E_A(\mathrm{scal}) = \mathrm{Res}_A\left(x_1\,\frac{\partial\,\mathrm{scal}}{\partial x_1},\ \dots,\ x_\ell\,\frac{\partial\,\mathrm{scal}}{\partial x_\ell}\right) = \prod_{F \text{ face of } P'} (\Delta_{F\cap A})^{\alpha_F}$$
- For given $\mathbf{b}$, $\mathbf{d}$, L, as $d_i > 0$,

$$\boxed{\begin{array}{c}\exists\,\mathbf{v} \in \mathbb{R}^\ell \setminus \{\mathbf{0}\},\\ \mathcal{F}_{\mathbf{v}} \text{ has solutions in } (\mathbb{C}^*)^\ell\end{array}} \Rightarrow \boxed{\begin{array}{c}\exists\, F \text{ face of } P',\\ \Delta_{F\cap A} = 0\end{array}} \Rightarrow \boxed{E_A(\mathrm{scal}) = 0}$$

If $\ell = 2$:

$$r_1(\mathbf{x}) = \frac{b_1}{2x_1} - \frac{1}{4d_1}\left(\frac{L_{111}}{x_1} + \frac{2L_{112}x_2}{x_1^2} + \frac{2L_{122}}{x_1} - \frac{L_{122}x_1}{x_2^2}\right)$$
$$r_2(\mathbf{x}) = \frac{b_2}{2x_2} - \frac{1}{4d_2}\left(\frac{L_{222}}{x_2} + \frac{2L_{122}x_1}{x_2^2} + \frac{2L_{112}}{x_2} - \frac{L_{112}x_2}{x_1^2}\right)$$

$$\mathrm{scal}(\mathbf{x}) = d_1\, r_1(\mathbf{x}) + d_2\, r_2(\mathbf{x})$$

$$E_A(\mathrm{scal}) = \begin{vmatrix} L_{122} & L'_{222} & L'_{111} & L_{112} & & \\ & L_{122} & L'_{222} & L'_{111} & L_{112} & \\ & & L_{122} & L'_{222} & L'_{111} & L_{112} \\ 3L_{122} & 2L'_{222} & L'_{111} & & & \\ & 3L_{122} & 2L'_{222} & L'_{111} & & \\ & & 3L_{122} & 2L'_{222} & L'_{111} & \end{vmatrix}$$

where

$$L'_{111} = L_{111} + 2L_{122} - 2b_1 d_1$$
$$L'_{222} = L_{222} + 2L_{112} - 2b_2 d_2$$

Q: For $\ell = 2$, finiteness holds if $d_i > 0$. Is the same true for $\ell \geq 3$?

Numeric solutions on full flag manifolds G/T

G	SU(3)	SU(4)	SU(5)	SU(6)	SO(5)	SO(7)	Sp(3)	SO(8)
ℓ	3	6	10	15	4	9	9	12
$D_{\ell-1}$	13	1 683	1 462 563	7.9×10^9	63	256 729	256 729	45 046 719
BKK Bound	4	80	9 168	6 603 008	12	5 376	5 232	239 744
# solutions in $(\mathbb{C}^*)^\ell$	4	59	7 908	5 037 448	10	4 224	4 512	150 256
# solutions in $(\mathbb{R}^*)^\ell$	4	29	1 596	191 252	6	750	728	11 128
# solutions in $\mathbb{R}_+^\ell$, i.e., # G-invariant Einstein metrics on G/H	4	29	396	6572	6	48	64	184
# isometry classes of G-invariant Einstein metrics on G/H	2	4	12	35	2	5	4	5

using `HomotopyContinuation.jl`

⚠ Except for G = SU(3), the BKK bound is never achieved, thus, these systems are *not* generic.

Q: Find more examples of G/H where BKK bound is achieved.