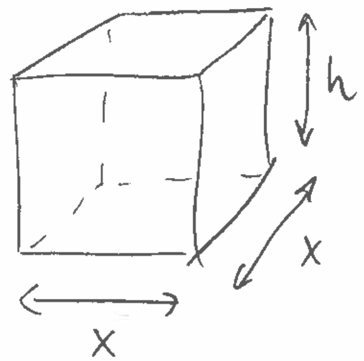
SOLUTION TO HW 8:

V = Volume of box
A = Surface area of box
x = length of sides of square base
h = height of box



$$V = \left(\begin{array}{c}\text{Area of}\\ \text{base}\end{array}\right) \cdot (\text{height}) = x^2 \cdot h$$

$$A = \underbrace{4xh}_{\left(\begin{array}{c}\text{4 side}\\ \text{faces}\end{array}\right)} + \underbrace{2x^2}_{\left(\begin{array}{c}\text{top \&}\\ \text{bottom}\end{array}\right)}$$

- Constraint: $A = 8 \Rightarrow 4xh + 2x^2 = 8$

$$\Rightarrow h = \frac{8 - 2x^2}{4x} = \frac{4 - x^2}{2x}$$

- substitute back in volume (target function):

$$V = x^2 h = x^2 \left(\frac{4 - x^2}{2x}\right) = \frac{x(4 - x^2)}{2} = 2x - \frac{x^3}{2}$$

$$V'(x) = 2 - \frac{3x^2}{2} = 0 \Leftrightarrow 3x^2 = 4$$

$$\Leftrightarrow \boxed{x = \frac{2}{\sqrt{3}}} \quad \left(\begin{array}{c}\text{only interested}\\ \text{in } x > 0 \text{ here}\end{array}\right)$$

- The only critical point is $x = \frac{2}{\sqrt{3}}$; let's check if it is a loc. maximum using Second Derivative Test:

$$V''(x) = -3x \Rightarrow V''\left(\frac{2}{\sqrt{3}}\right) = -3 \cdot \frac{2}{\sqrt{3}} < 0 \Rightarrow \frac{\text{local}}{\text{max}}$$

Since $x = \frac{2}{\sqrt{3}}$ is the only critical point of $V(x) = 2x - \frac{x^3}{2}$ and it is a local max, this is also the global max of $V(x)$. Therefore, the maximum volume that a box with square base built out of 8 ft^2 of carboard can have is

$$V\left(\frac{2}{\sqrt{3}}\right) = 2 \cdot \frac{2}{\sqrt{3}} - \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^3$$

$$= \frac{4}{\sqrt{3}} - \frac{4}{3\sqrt{3}} = \frac{4}{\sqrt{3}} \cdot \left(1 - \frac{1}{3}\right) = \frac{4}{\sqrt{3}} \cdot \frac{2}{3}$$

$$= \frac{8}{3\sqrt{3}} \ ft^3.$$

Bonus: Note this maximum volume is achieved exactly when the box is a perfect cube (i.e., $x = h$).

Indeed:

$$h = \frac{4 - x^2}{2x} = \frac{4 - \left(\frac{2}{\sqrt{3}}\right)^2}{2 \cdot \frac{2}{\sqrt{3}}} = \frac{4 - \frac{4}{3}}{\frac{4}{\sqrt{3}}} = \frac{\frac{8}{3}}{\frac{4}{\sqrt{3}}} = \frac{8}{3} \cdot \frac{\sqrt{3}}{4}$$

$$= \frac{2\sqrt{3}}{3} = \frac{2}{\sqrt{3}} = x.$$

and $V = \frac{8}{3\sqrt{3}} = \left(\frac{2}{\sqrt{3}}\right)^3$ for a cube whose sides measure $\frac{2}{\sqrt{3}}$.