Name: ANSWERS Lehman ID: ____________

MAT 175
Midterm 2
December 5, 2018

Instructions:
Turn off and put away your cell phone.
Please write your Name and Lehman ID # on the top of this page.
Please sign and date the pledge below to comply with the Code of Academic Integrity.
No consultation material, calculators, or electronic devices are allowed during the exam.
If any question is unclear, raise your hand to ask for clarifications.
The regular amount of time you have to complete the exam is 100 minutes.
You must show all of your work! *No credit will be given for unsupported answers.*
Please try to be as organized, objective, and logical as possible in your answers.

#	Points	Score
1	20	
2	~~10~~ 20	
3	10	
4	10	
~~5~~	~~10~~	CANCELLED
6	10	
7	10	
8	10	
9	10	
Total	100	

My signature below certifies that I complied with the CUNY Academic Integrity Policy and the Lehman College Code of Academic Integrity in completing this examination.

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Signature Date

Problem 1 (20 pts): Consider the function $f(x) = x^3 - 3x^2 - 9x + 3$

a) **(5 pts)** Find all the critical points of $f(x)$.

$f'(x) = 3x^2 - 6x - 9$ (exists at all points)

$f'(x) = 0 \iff x^2 - 2x - 3 = 0$

$\iff \underline{x = -1}$ and $\underline{x = 3}$

$\boxed{\text{Critical points: } x = -1 \text{ and } x = 3}$

b) **(5 pts)** Determine whether each critical point found above is a local minimum, a local maximum, or neither.

$f''(x) = 6x - 6$

$f''(-1) = -6 - 6 = -12 < 0 \Rightarrow$ $\boxed{x = -1 \text{ is a local maximum}}$

$f''(3) = 18 - 6 = 12 > 0 \Rightarrow$ $\boxed{x = 3 \text{ is a local minimum}}$

c) **(5 pts)** What are the global minimum and global maximum of $f(x)$ on the interval $I = [-3, 3]$?

Values at endpoints:

$$f(-3) = (-3)^3 - 3(-3)^2 - 9(-3) + 3$$

$$= -27 - \cancel{27} + \cancel{27} + 3 = -24$$

$$f(3) = 27 - \cancel{27} - 9 \cdot 3 + 3 = -24$$

$\leftarrow$ Global min

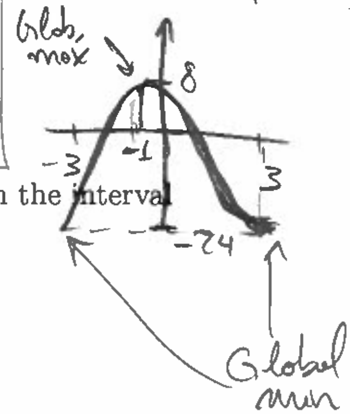
Values at crit. pts. in the interior:

$$f(-1) = (-1)^3 - 3 \cdot 1 + 9 + 3 = -4 + 12 = 8 \leftarrow \text{Global max}$$

Thus:

$$\boxed{\begin{array}{l} \min\limits_{x\in[-3,3]} f(x) = \underline{-24} = f(-3) = f(3) \\ \max\limits_{x\in[-3,3]} f(x) = \underline{8} = f(-1) \end{array}}$$

$\leftarrow$ (Achieved at two points!)



d) **(5 pts)** What are the global minimum and global maximum of $f(x)$ on the interval $I = [0, 6]$?

Values at endpoints:

$$f(0) = \underline{3}$$

$$f(6) = 216 - 3 \cdot 36 - 9 \cdot 6 + 3$$

$$= 216 - 108 - 51 = \underline{57} \leftarrow \text{Global max}$$

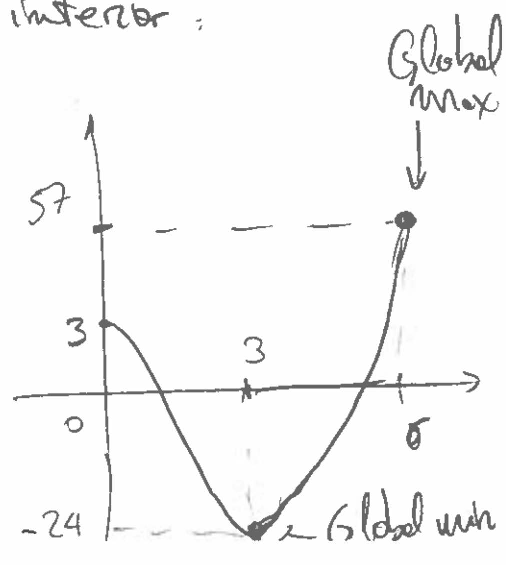
Values at critical points in the interior:

$$f(3) = \underline{-24} \quad (\text{from (c)})$$

$\nwarrow$ Global min

Thus:

$$\boxed{\begin{array}{l} \min\limits_{x\in[0,6]} f(x) = -24 = f(3) \\ \max\limits_{x\in[0,6]} f(x) = 57 = f(6) \end{array}}$$



Problem 2 (20 pts): $f(x) = (x-1)^2 e^x$

a) (10 pts) Find <u>all the critical points</u> of the function [illegible], and determine if they are <u>local minima</u>, <u>local maxima</u>, or <u>neither</u>.

$$f'(x) = 2(x-1)e^x + (x-1)^2 e^x = (x^2 - \cancel{2x} + 1 + \cancel{2x} - 2)e^x$$

$$f'(x) = (x^2-1)e^x \quad \text{(exists at all points)}$$

$$f'(x) = 0 \iff x = \pm 1 \qquad \boxed{\text{critical points: } \underline{x=-1, x=1}}$$

$$f''(x) = 2xe^x + (x^2-1)e^x = (x^2+2x-1)e^x$$

$$f''(-1) = (1-2-1)e^{-1} = -\frac{2}{e} < 0 \overset{\text{(2nd der. test)}}{\Longrightarrow} \boxed{x=-1 \text{ is a } \underline{\text{loc. max.}}}$$

$$f''(1) = (1+2-1)e = 2e > 0 \overset{\text{(2nd der. test)}}{\Longrightarrow} \boxed{x=1 \text{ is a } \underline{\text{local. min.}}}$$

b) (10 pts) Find <u>all the critical points</u> of the function $f(x) = 3x^{2/3} - x$, and determine if they are <u>local minima</u>, <u>local maxima</u>, or <u>neither</u>.

$$f'(x) = 3 \cdot \frac{2}{3} x^{-1/3} - 1 = 2x^{-1/3} - 1$$

<u>Note</u>: $f'(x)$ does not exist if $x=0$!

If $x \neq 0$, $f'(x) = 0 \iff \frac{2}{\sqrt[3]{x}} = 1 \iff \sqrt[3]{x} = 2 \iff x = 8$

$$\boxed{\text{Critical points: } \underline{x=0} \text{ and } \underline{x=8}}$$

$$f''(x) = 2\left(-\frac{1}{3}\right)x^{-4/3} = -\frac{2}{3}x^{-4/3}$$

$$f''(8) = -\frac{2}{3} 8^{-4/3} = -\frac{2}{3} \cdot 2^{-4} < 0 \overset{\text{2nd der. test}}{\Longrightarrow} \boxed{x=8 \text{ is a } \underline{\text{local max.}}}$$

$$\left.\begin{array}{l} f'(x) < 0 \text{ if } x<0 \text{ near } x=0 \\ f'(x) > 0 \text{ if } x>0 \text{ near } x=0 \end{array}\right\} \overset{\text{1st der. test}}{\Longrightarrow} \boxed{x=0 \text{ is a } \underline{\text{local min}}}$$

Problem 3 (10 pts): Consider the Initial Value Problem below:

$$\begin{cases} \dfrac{dy}{dx} = x^2 - \sin(2x) \\ y(0) = 5 \end{cases}$$

a) **(7 pts)** Find the solution $y(x)$ to the above Initial Value Problem.

$$y(x) = \int x^2 - \sin 2x \, dx = \frac{x^3}{3} + \frac{\cos 2x}{2} + C$$

$$5 = y(0) = \frac{1}{2} + C \Rightarrow C = 5 - \frac{1}{2} = \frac{9}{2}.$$

$$\boxed{y(x) = \frac{x^3}{3} + \frac{\cos 2x}{2} + \frac{9}{2}}$$

b) **(3 pts)** Is the solution $y(x)$ concave up or concave down when $1 < x < +\infty$?

$$y''(x) = \frac{d}{dx}\left(x^2 - \sin 2x\right) = 2x - 2\cos 2x = 2(x - \cos 2x)$$

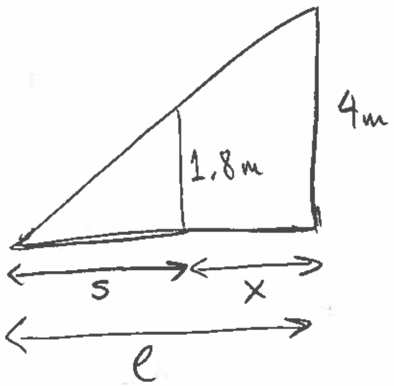
For all $x \in \mathbb{R}$, $-1 \leq \cos 2x \leq 1$, so if $x > 1$, then

$x - \underbrace{\cos 2x}_{\in [-1,1]} > 0$, hence $y''(x) = 2(x - \cos 2x) > 0$

Thus, $\boxed{y(x) \text{ is concave up when } 1 < x < +\infty}$

Problem 4 (10 pts): A person is 1.8 m tall and walks at speed 1 m/s towards a streetlight which is 4 m above the ground.

a) **(7 pts)** At what speed is the tip of the person's shadow moving?



x = distance from person to streetlight

s = length of shadow

ℓ = distance from tip of shadow to streetlight

Know: $\frac{dx}{dt} = -1 \frac{m}{s}$ Want: $\frac{d\ell}{dt} = ?$

$$\frac{s}{1.8} = \frac{\ell}{4} \Rightarrow \frac{\ell - x}{1.8} = \frac{\ell}{4} \Rightarrow \ell - x = \frac{18}{40}\ell = \frac{9}{20}\ell$$

$$\Rightarrow \frac{11}{20}\ell = x \Rightarrow \frac{11}{20}\frac{d\ell}{dt} = \frac{dx}{dt} = -1 \Rightarrow \boxed{\frac{d\ell}{dt} = -\frac{20}{11} \text{ m/s}}$$

A: The tip of the person's shadow is moving at $\frac{20}{11}$ m/s.

b) **(3 pts)** At what rate is the person's shadow shortening?

$$s = \frac{1.8}{4}\ell = \frac{9}{20}\ell \Rightarrow \frac{ds}{dt} = \frac{9}{20}\frac{d\ell}{dt} = \frac{9}{20}\left(-\frac{20}{11}\right)$$

$$\Rightarrow \boxed{\frac{ds}{dt} = -\frac{9}{11} \text{ m/s}}$$

A: The shadow is shortening at $\frac{9}{11}$ m/s

(CANCELLED)

Problem 5 (10 pts): Consider the function $f(t) = t^2 + \ln t^2$, defined for all $t \neq 0$.

a) **(7 pts)** Determine where $f(t)$ is <u>increasing</u> and where it is <u>decreasing</u>.

$$f'(t) = 2t + \frac{1}{t^2} \cdot 2t = 2t + \frac{2}{t} = 2\left(t + \frac{1}{t}\right)$$

$$f'(t) = 2\,\frac{t^2+1}{t} \quad {>0}$$

So $f'(t) > 0 \iff t > 0$

$f'(t) < 0 \iff t < 0$

A: $f(t)$ is <u>increasing</u> on $(0, +\infty)$
$f(t)$ is <u>decreasing</u> on $(-\infty, 0)$

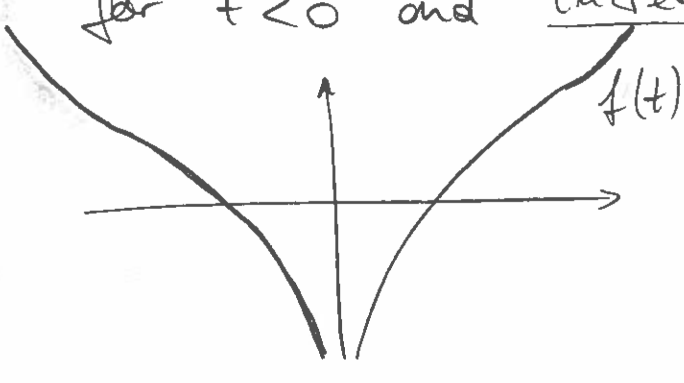
b) **(3 pts)** <u>Use a limit</u> to show that $f(t)$ <u>does not</u> have a global minimum.

Note that $f(t)$ is <u>not defined</u> at $t = 0$,

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} t^2 + \ln t^2 = \lim_{u \to 0_+} \underbrace{u}_{\to 0} + \underbrace{\ln u}_{\to -\infty} = -\infty$$

$u = t^2$

Since $\boxed{\lim_{t \to 0} f(t) = -\infty}$, $f(t)$ does not have a global minimum, despite the fact it is <u>decreasing</u> for $t < 0$ and <u>increasing</u> for $t > 0$:



Problem 5 [redacted] **(10 pts):** Evaluate the following definite integrals:

a) **(5 pts)** $\int_0^3 x^2 + 4x - e^x \, dx$

$$\int_0^3 x^2 + 4x - e^x \, dx = \frac{x^3}{3} + 2x^2 - e^x \Big|_0^3$$

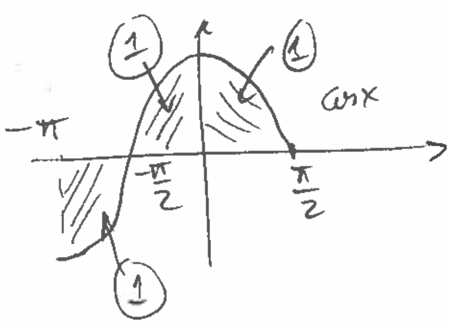
$$= \left(\frac{27}{3} + 2.9 - e^3\right) + 1$$

$$= 27 - e^3 + 1 = \boxed{28 - e^3}$$

b) **(5 pts)** $\int_{-\pi}^{\pi/2} \sqrt{3} \cos x \, dx$

$$\int_{-\pi}^{\pi/2} \sqrt{3} \cos x \, dx = \sqrt{3} \sin x \Big|_{-\pi}^{\pi/2} = \sqrt{3}.(1 - 0) = \boxed{\sqrt{3}}$$

Also, note:

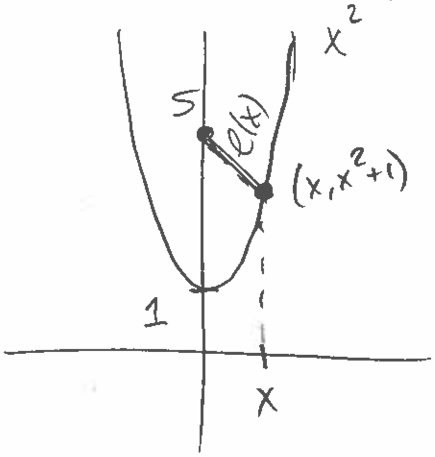


$$\int_{-\pi}^{\pi/2} \cos x \, dx = -1 + 1 + 1 = 1$$

$$\text{so } \sqrt{3} \int_{-\pi}^{\pi/2} \cos x \, dx = \sqrt{3}$$

"Each half lump has area 1"...

Problem 6 (10 pts): Find the points on the curve $y = x^2 + 1$ at minimum distance from the point $(0, 5)$.



Point on $y = x^2+1$: $\quad (x, x^2+1)$

Distance from $(0,5)$ to (x, x^2+1):

$$\ell(x) = \sqrt{(x-0)^2 + (x^2+1-5)^2}$$

$$\ell(x) = \sqrt{x^2 + (x^2-4)^2}$$

$$\ell'(x) = \frac{1}{2\sqrt{x^2+(x^2-4)^2}} \cdot \left(2x + 2(x^2-4) \cdot 2x\right)$$

Note: This denominator never vanishes, so $\ell'(x)$ always exists!

(Note: It is equivalent to minimize the square distance $\ell(x)^2$, which leads directly here...)

$$= \frac{x + 2x(x^2-4)}{\sqrt{x^2+(x^2-4)^2}} = 0 \iff x + 2x(x^2-4) = 0$$

$$\iff x + 2x^3 - 8x = 0 \iff 2x^3 - 7x = 0$$

$$\iff x = 0 \text{ or } x \neq 0 \text{ and } 2x^2 = 7, \text{ i.e., } x = \pm\sqrt{\tfrac{7}{2}}$$

<u>Critical points</u>: $x = \pm\sqrt{\frac{7}{2}}$ and $x = 0$.

If $x = 0$: $\quad (x, x^2+1) = (0, 1)$

Distance to $(0,5)$: $\ell(0) = \underline{\underline{4}}$

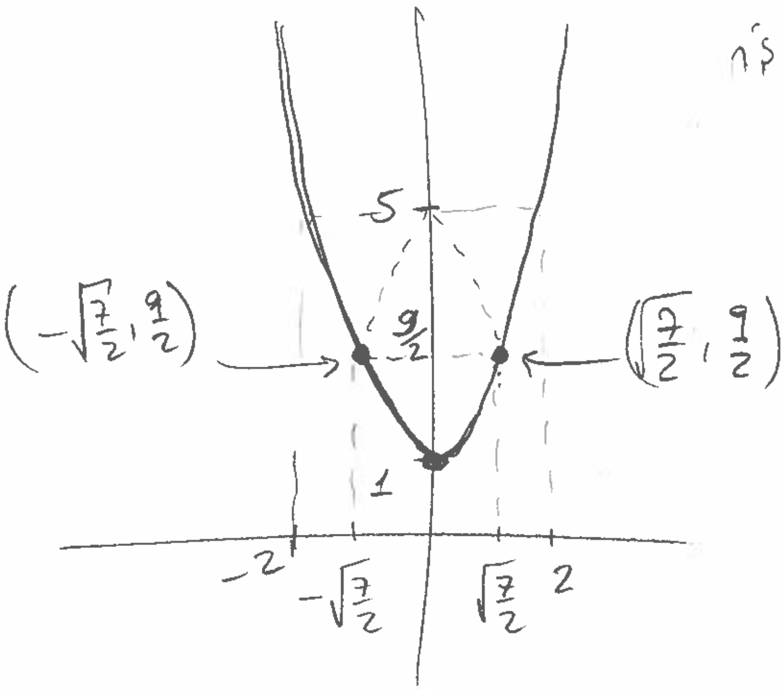
If $x = \pm\sqrt{\frac{7}{2}}$ $\quad (x, x^2+1) = \left(\pm\sqrt{\frac{7}{2}}, \frac{9}{2}\right)$

Distance to $(0,5)$: $\ell\left(\pm\sqrt{\frac{7}{2}}\right) = \sqrt{\frac{7}{2} + \left(\frac{7}{2} - 4\right)^2}$

$$= \sqrt{\frac{7}{2} + \frac{1}{4}} = \sqrt{\frac{15}{4}} = \boxed{\frac{\sqrt{15}}{2}}$$

$\longrightarrow$

Since $\frac{\sqrt{15}}{2} < 4$, the minimum distance is achieved at the points $\boxed{\left(\pm\sqrt{\frac{7}{2}}, \frac{9}{2}\right)}$ and this minimum distance is $\ell\left(\pm\sqrt{\frac{7}{2}}\right) = \boxed{\frac{\sqrt{15}}{2}}$



Problem 7 (10 pts): Consider the function $F(x) = \int_0^{x^2+1} e^{-t^3}\,dt$.

a) **(5 pts)** Compute $F'(1)$

Let $G(u) = \int_0^u e^{-t^3}\,dt$. Then $G'(u) = e^{-u^3}$, by the 2nd Fund. Thm. of Calculus. By the Chain Rule:

$$F(x) = G(x^3) \Rightarrow F'(x) = \frac{dG}{du} \cdot \frac{du}{dx}$$

$$u = x^2 + 1 \qquad = e^{-u^3} \cdot 2x = e^{-(x^2+1)^3} \cdot 2x$$

So $F'(1) = e^{-2^3} \cdot 2 = 2e^{-8} = \boxed{\frac{2}{e^8}}$

b) **(5 pts)** Compute $F''(1)$

$$F''(x) = \frac{d}{dx} F'(x) = \frac{d}{dx} e^{-(x^2+1)^3} \cdot 2x$$

$$= e^{-(x^2+1)^3} (-3)(x^2+1)^2 \cdot 2x \cdot 2x + e^{-(x^2+1)^3} \cdot 2$$

$$= -12x^2(x^2+1)^2 e^{-(x^2+1)^3} + 2e^{-(x^2+1)^3}$$

$$= 2e^{-(x^2+1)^3}\left(1 - 6x^2(x^2+1)^2\right)$$

So $F''(1) = 2e^{-8}(1 - 6 \cdot 4) = \frac{2}{e^8}(-23) = \boxed{-\frac{46}{e^8}}$

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Problem [redacted] **(10 pts):** Find the area under the graph of $f(x) = x^3\cos(x^4)$ between $x = 0$ and $x = \left(\frac{\pi}{2}\right)^{1/4}$.

$$\int x^3 \cos(x^4)\, dx = \int \cos(u)\, \frac{du}{4} = \frac{1}{4}\sin u + C$$

$$u = x^4$$

$$du = 4x^3\, dx$$

$$= \frac{1}{4}\sin(x^4) + C.$$

$$\Rightarrow \int_0^{(\pi/2)^{1/4}} x^3\cos(x^4)\, dx = \left.\frac{\sin(x^4)}{4}\right|_0^{\left(\frac{\pi}{2}\right)^{1/4}}$$

$$= \frac{\sin\left(\frac{\pi}{2}\right)}{4} - 0 = \boxed{\frac{1}{4}}$$

A: The area under the graph of $f(x) = x^3\cos(x^4)$ between $x=0$ and $x = \left(\frac{\pi}{2}\right)^{1/4}$ is $\boxed{\frac{1}{4}}$