SOLUTION TO PRACTICE PROBLEMS FOR MIDTERM 1

1. a) $$\lim_{x\to 0} \frac{2\sin 4x}{3x} = 2 \cdot \lim_{x\to 0} \overbrace{\frac{\sin 4x}{4x}}^{1} \frac{4}{3} = \frac{8}{3}.$$

b) $$\lim_{x\to \frac{\pi}{2}} \frac{2\sin 4x}{3x} = \frac{2\sin 2\pi}{\frac{3\pi}{2}} = 0.$$

c) $$\lim_{x\to 2} \frac{x^2+3x-10}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+5)}{\cancel{x-2}} = 7.$$

d) $$\lim_{x\to 2} \frac{x^2-4x+8}{x-2}$$ D.N.E. b/c $$\lim_{x\to 2_-} \frac{x^2-4x+8}{x-2} = -\infty$$ $$\lim_{x\to 2_+} \frac{x^2-4x+8}{x-2} = +\infty$$

e) $$\lim_{a\to 3} \frac{\sqrt{a+1}-2}{a+3} = \frac{\sqrt{4}-2}{6} = 0.$$

f) $$\lim_{a\to 3} \frac{\sqrt{a+1}-2}{a-3} = \lim_{a\to 3} \frac{(\sqrt{a+1}-2)}{(a-3)} \frac{(\sqrt{a+1}+2)}{(\sqrt{a+1}+2)} =$$

$$= \lim_{a\to 0} \frac{a+1-4}{(a-3)(\sqrt{a+1}+2)} = \lim_{a\to 0} \frac{1}{\sqrt{a+1}+2} = \frac{1}{4}.$$

g) $$\lim_{t\to 0} \underbrace{t^2}_{\text{goes to zero}} \underbrace{\cos\left(\frac{\pi}{t}\right)}_{\text{bounded}} = 0$$ by the Squeeze Theorem.

2\. a) $\lim\limits_{x \to +\infty} \frac{\sin x}{x} = 0$ b/c $|\sin x| \leq 1$ is bounded.

b) $\lim\limits_{x \to +\infty} \frac{x^3+2x+1}{x^2-1} = \lim\limits_{x \to +\infty} \frac{x\,\cancel{x^3}\left(1+\frac{2}{x^2}+\frac{1}{x^3}\right)}{\cancel{x^2}\left(1-\frac{1}{x^2}\right)} = +\infty$

c) $\lim\limits_{x \to -\infty} \frac{x^3+2x+1}{x^2-1} = \lim\limits_{x \to -\infty} \frac{x\,\cancel{x^3}\left(1+\frac{2}{x^2}+\frac{1}{x^3}\right)}{\cancel{x^2}\left(1-\frac{1}{x^2}\right)} = -\infty$.

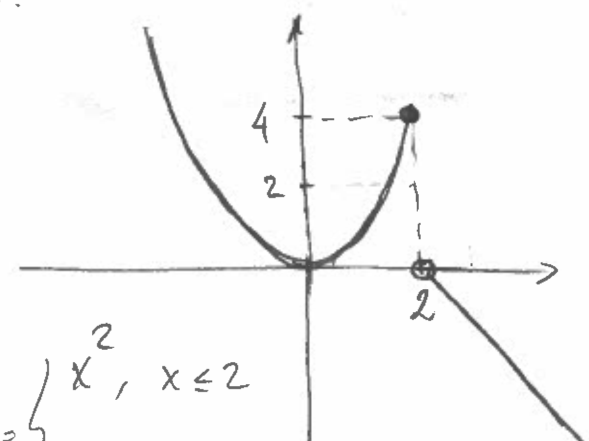
d) $\lim\limits_{x \to +\infty} \frac{3x^5+x^3+x+1}{4x^5-x^2-8} = \lim\limits_{x \to +\infty} \frac{\cancel{x^5}\left(3+\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^5}\right)}{\cancel{x^5}\left(4-\frac{1}{x^3}-\frac{8}{x^3}\right)} = \frac{3}{4}$

e) $\lim\limits_{x \to -\infty} \frac{3x^5+x^3+x+1}{4x^5-x^2-8} = \lim\limits_{x \to -\infty} \frac{\cancel{x^5}\left(3+\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^5}\right)}{\cancel{x^5}\left(4-\frac{1}{x^3}-\frac{8}{x^3}\right)} = \frac{3}{4}$.

f) $\lim\limits_{x \to +\infty} \frac{x^3+2x^2+1}{-2x^7+3x^4+x^2} = \lim\limits_{x \to +\infty} \frac{\cancel{x^3}\left(1+\frac{2}{x}+\frac{1}{x^3}\right)}{x^4\,\cancel{x^7}\left(-2+\frac{3}{x^4}+\frac{1}{x^5}\right)} = 0$

g) $\lim\limits_{x \to -\infty} \frac{x^3+2x^2+1}{-2x^7+3x^4+x^2} = \lim\limits_{x \to -\infty} \frac{\cancel{x^3}\left(1+\frac{2}{x}+\frac{1}{x^3}\right)}{x^4\,\cancel{x^7}\left(-2+\frac{3}{x^3}+\frac{1}{x^5}\right)} = 0$

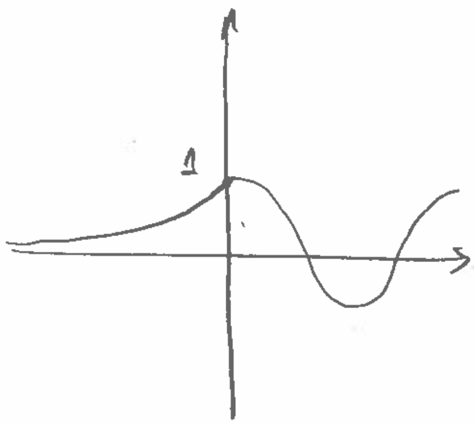
3\.



$f(x) = \begin{cases} x^2, & x \leq 2 \\ 2-x, & x > 2 \end{cases}$

$f(x)$ is continuous at all points except for $x=2$, where it is discontinuous.

4.



$$f(x) = \begin{cases} e^x & x \le 0 \\ \cos x & x > 0 \end{cases}$$

$f(x)$ is continuous at all points.

5. The function

$$f(x) = \begin{cases} \dfrac{2\sin(ax)}{x} & \text{if } x \le 0 \\ x + a^2 + 1 & \text{if } x > 0 \end{cases}$$

is continuous at all points $x \neq 0$. In order for it to be continuous at $x = 0$, it is necessary (and sufficient) that the lateral limits $x \to 0_+$ and $x \to 0_-$ agree:

$$\lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} x + a^2 + 1 = a^2 + 1$$

$$\lim_{x \to 0_-} f(x) = \lim_{x \to 0_-} \frac{2\sin(ax)}{x} = 2a \underbrace{\lim_{x \to 0_-} \frac{\sin(ax)}{ax}}_{1} = 2a$$

$$\Rightarrow \quad a^2 + 1 = 2a \iff a^2 - 2a + 1 = 0$$

$$\iff (a-1)^2 = 0$$

$$\iff a = 1.$$

Thus, if $a = 1$, then $f(x)$ is continuous at all points.

6. $f(x) = 2x+3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)+3 - (2x+3)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x} + 2h + \cancel{3} - \cancel{2x} - \cancel{3}}{h} = \lim_{h\to 0} \frac{2h}{h} = 2.$$

7. $g(x) = 5x^2 - x$

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{5(x+h)^2 - (x+h) - (5x^2 - x)}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{5x^2} + 10xh + 5h^2 - \cancel{x} - h - \cancel{5x^2} + \cancel{x}}{h} = \lim_{h\to 0} \frac{10xh + 5h^2 - h}{h}$$

$$= \lim_{h\to 0} 10x - 1 + 5h = 10x - 1.$$

8. a) $f(x) = 1 + 3\pi x^4 - 2x + e^x$

$$f'(x) = 3\pi \cdot 4x^3 - 2 + e^x = 12\pi x^3 - 2 + e^x$$

b) $F(x) = \frac{2}{x} - \sqrt{5}\,x - 5\sqrt{x} + 5\cos(x-1)$

$$F'(x) = -\frac{2}{x^2} - \sqrt{5} - \frac{5}{2\sqrt{x}} - 5\sin(x-1)$$

c) $g(t) = \frac{1}{\sqrt{2t+2}} - te^{2t+1} - \tan\left(4t^2 + \frac{\pi}{4}\right)$

$$g'(t) = -\frac{1}{2}(2t+2)^{-3/2} \cdot 2 - \left(e^{2t+1} + te^{2t+1} \cdot 2\right) - \sec^2\left(4t^2 + \frac{\pi}{4}\right)(8t)$$

$$= -\frac{1}{(2t+2)^{3/2}} - e^{2t+1} - 2te^{2t+1} - 8t\sec^2\left(4t^2 + \frac{\pi}{4}\right).$$

d) $A(\theta) = 3\sin\theta\cos\theta + e^{\theta}\cos 5\theta$

$A'(\theta) = 3\cos^2\theta - 3\sin^2\theta + e^{\theta}\cos 5\theta + e^{\theta}(-\sin 5\theta).5$

e) $g(s) = \dfrac{e^{s^2+1} - \sin(\sqrt{s})}{s^2+1}$

$g'(s) = \dfrac{\left(e^{s^2+1}.(2s) - \cos(\sqrt{s}).\frac{1}{2\sqrt{s}}\right)(s^2+1) - \left(e^{s^2+1} - \sin(\sqrt{s})\right)2s}{(s^2+1)^2}$

? Using the computations of first derivatives in the previous exercise:

a) $f'(0) = -2+1 = -1$

$y - 2 = -1(x-0) \implies \boxed{y = -x+2}$

b) $F'(1) = -2 - \sqrt{5} - \frac{5}{2} - 5\underbrace{\sin(0)}_{=0} = -\frac{9}{2} - \sqrt{5}$

$y - (2-\sqrt{5}) = \left(-\frac{9}{2} - \sqrt{5}\right)(x-1)$

$\implies y = \left(-\frac{9}{2} - \sqrt{5}\right)x + 2 - \cancel{\sqrt{5}} + \frac{9}{2} + \cancel{\sqrt{5}}$

$\implies \boxed{y = \left(-\frac{9}{2} - \sqrt{5}\right)x + \frac{13}{2}}$

c) $g'(0) = -\frac{1}{2^{3/2}} - e = -\frac{1}{2\sqrt{2}} - e$

$y - \left(\frac{1}{\sqrt{2}} - 1\right) = \left(-\frac{1}{2\sqrt{2}} - e\right)(x-0) \implies \boxed{y = \left(-\frac{1}{2\sqrt{2}} - e\right)x + \frac{1}{\sqrt{2}} - 1}$

d) $A'(0) = 3\cos(0) - e^0 \cos 0 = 3 - 1 = 2$

$y - 1 = 2(x-0) \Rightarrow \boxed{y = 2x + 1}$

e) $g'(s)$ is not well-defined at $s=0$, because there is a term $\frac{1}{2\sqrt{s}}$ which comes from differentiating $\sin(\sqrt{s})$, a function which is only differentiable for $s>0$.

Thus, there is no tangent line at $(0,e)$. [My apologies – this was not intentional...]

10. $s(t) = t^2 - 2\sqrt{t^2+1} + e^t$ position

$$s'(t) = 2t - \frac{2}{2\sqrt{t^2+1}} \cdot 2t + e^t$$

$$= 2t - \frac{2t}{\sqrt{t^2+1}} + e^t$$ is the velocity at time t

$$s''(t) = 2 - \frac{2\sqrt{t^2+1} - 2t \cdot \frac{1}{2\sqrt{t^2+1}} \cdot 2t}{t^2+1} + e^t$$

$$= 2 - \frac{\left(2\sqrt{t^2+1} - \frac{2t^2}{\sqrt{t^2+1}}\right)}{t^2+1} + e^t$$ is the acceleration at time t.

Thus, at time $t=1$,

- the velocity is $s'(1) = 2 - \frac{2}{\sqrt{2}} + e = \underline{\underline{2-\sqrt{2}+e}}$
- the acceleration is $s''(1) = 2 - \left(\frac{2\sqrt{2} - \frac{2}{\sqrt{2}}}{2}\right) + e = \underline{\underline{2 - \frac{\sqrt{2}}{2} + e}}$

11. $s(t) = -16t^2 + 16t + 32$ is position (height) at time t

The object hits the ground when $s(t) = 0$, i.e., when the height is 0.

$$s(t) = -16t^2 + 16t + 32 = 0 \iff 16(-t^2 + t + 2) = 0$$
$$\iff t^2 - t - 2 = 0$$
$$\iff \boxed{t = 2} \quad \text{or} \quad t = -1$$

Velocity at time t is:

$$s'(t) = -32t + 16$$

Acceleration at time t is:

$$s''(t) = -32$$

Thus, the velocity and acceleration of the object when it hits the ground are, respectively,

$$s'(2) = -64 + 16 = -48 \quad \text{and} \quad s''(2) = -32$$

12. $x^2 + y^3 + y = 1$, $y = y(x)$ near $(1,0)$

Implicitly differentiating,

$$2x + 3y^2 y' + y' = 0 \implies y' \cdot (1 + 3y^2) = -2x$$
$$\implies y'(x) = -\frac{2x}{1 + 3y^2}$$

The slope of the tangent line at $(1,0)$ is therefore

$m = -\dfrac{2 \cdot 1}{1 + 3 \cdot 0^2} = \underline{-2}$; and the tangent line is hence

$$y - 0 = -2(x - 1) \implies \boxed{y = -2x + 2}$$

13. $f(x) = x^3 + x - 2$, $g(x) = f^{-1}(x)$.

$$f(g(x)) = x \implies f'(g(x)) \cdot g'(x) = 1$$

$$\implies g'(x) = \frac{1}{f'(g(x))}$$

$f'(x) = 3x^2 + 1$ $\implies g'(x) = \dfrac{1}{3g(x)^2 + 1}$

$f''(x) = 6x$

Clearly, $f(1) = 1^3 + 1 - 2 = 0$, so $g(0) = 1$. Thus

$$g'(0) = \frac{1}{3g(0)^2 + 1} = \frac{1}{3.1 + 1} = \boxed{\frac{1}{4}}$$

Differentiating $f'(g(x))\, g'(x) = 1$ once again, we have:

$$f''(g(x))\, g'(x)^2 + f'(g(x))\, g''(x) = 0$$

Evaluating at $x = 0$; using again that $g(0) = 1$ and $g'(0) = \frac{1}{4}$;

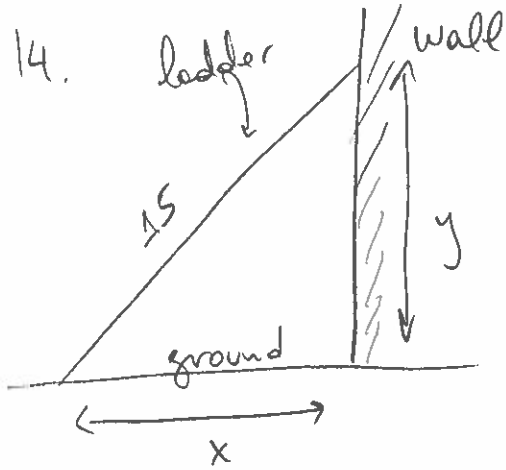
$$f''(g(0))\, g'(0)^2 + f'(g(0))\, g''(0) = 0$$

$$f''(1) \cdot \left(\tfrac{1}{4}\right)^2 + f'(1) \cdot g''(0) = 0$$

$$6 \cdot \frac{1}{16} + 4 \cdot g''(0) = 0 \to 4g''(0) = -\frac{3}{8}$$

$$\implies \boxed{g''(0) = -\frac{3}{32}}$$

14.



$x =$ dist. from foot of ladder to wall

$y =$ height where ladder rests on wall

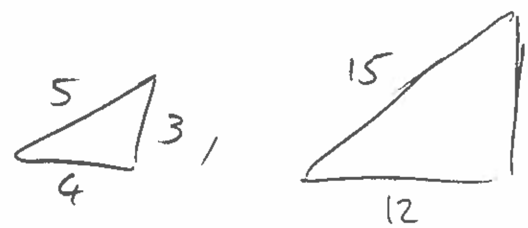
$$x^2 + y^2 = 15^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

We know: $\frac{dx}{dt} = +3$ ft/s, want $\frac{dy}{dt}$ when $x = 12$.

when $x = 12$, $y = 9$ b/c $12^2 + 9^2 = 15^2$.

(also a "3,4,5"-triangle



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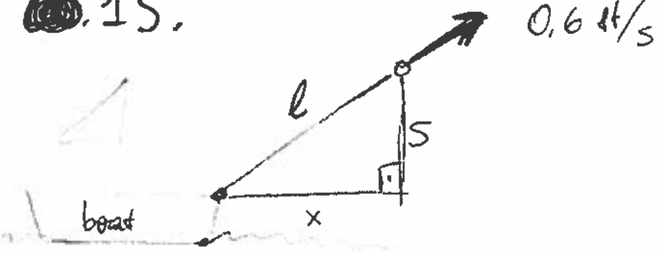
so:

$$2 \cdot 12 \cdot (+3) + 2 \cdot 9 \cdot \frac{dy}{dt} = 0$$

$$+36 + 18\frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{36}{18} = -2$$

A: The top of the ladder is sliding down at $\underline{2 \text{ ft/s}}$.

15.



l = length of rope out
x = distance from boat to dock.

Pythagoras: $x^2 + 5^2 = l^2$ $\qquad \frac{dl}{dt} = -0.6 \frac{ft}{s}$

$\Rightarrow x^2 + 25 = l^2$

Differentiate w.r.t. t: $\quad 2x \frac{dx}{dt} = 2l \frac{dl}{dt}$

Want: $\frac{dx}{dt}$ when $l = 13$:

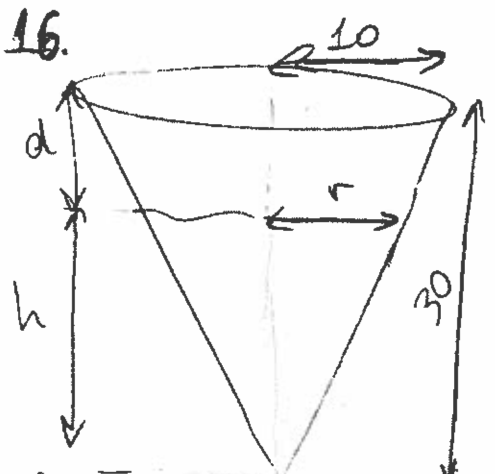
$l = 13 \Rightarrow x^2 + 25 = 13^2 = 169 \Rightarrow x^2 = 144 \Rightarrow x = 12$
$(x > 0)$

$2 \cdot 12 \cdot \frac{dx}{dt} = 2 \cdot 13 \cdot (-0.6) \Rightarrow \frac{dx}{dt} = -\frac{6}{10} \cdot \frac{13}{12}$

$\frac{dx}{dt} = -\frac{13}{20}$.

A: The boat is approaching the dock at speed $\frac{13}{20} \frac{ft}{s}$.

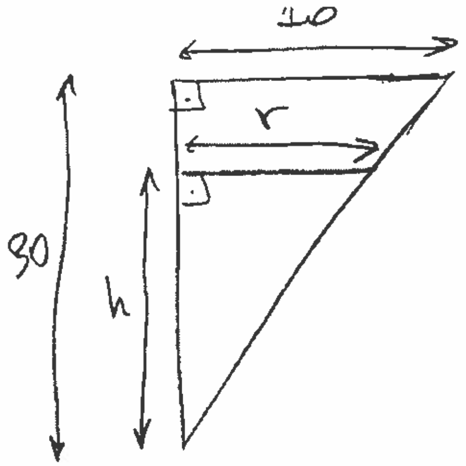
16.



h = height of water in tank
d = depth $= 30 - d$
V = Volume of water in tank.
r = radius of water level

$V = \frac{1}{3} \pi r^2 \cdot h$ $\qquad$ Want: $\frac{dh}{dt} = ?$ when $d = 3$.

$h + d = 30$



By similar triangles:

$$\frac{r}{h} = \frac{10}{30} = \frac{1}{3} \Rightarrow \boxed{r = \frac{h}{3}}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{3}\frac{dh}{dt} \quad (*)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\, 2r \frac{dr}{dt} h + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

$$\overset{(*)}{=} \frac{2\pi}{3} r \frac{1}{3}\frac{dh}{dt} h + \frac{\pi}{3} r^2 \frac{dh}{dt}$$

$$= \frac{2\pi r h}{9}\frac{dh}{dt} + \frac{\pi r^2}{3}\frac{dh}{dt}$$

When $d = 3$, $h = 30 - 3 = 27$, $r = \frac{h}{3} = \frac{27}{3} = 9$

$$\Rightarrow \underbrace{\frac{dV}{dt}}_{10} = \frac{2\pi . \cancel{9} . 27}{\cancel{9}} \cdot \frac{dh}{dt} + \frac{\pi . 9^2}{3}\frac{dh}{dt}$$

From problem

$$10 = 54\pi \frac{dh}{dt} + 27\pi \frac{dh}{dt} = 81\pi \frac{dh}{dt}$$

$$\Rightarrow \boxed{\frac{dh}{dt} = \frac{10}{81\pi}\ \frac{cm}{s}}$$

A: The water level is rising at a rate $\frac{10}{81\pi}$ cm/s.