Name: ANSWERS Lehman ID: ____________

**MAT 175**
**Midterm 1**
**April 1, 2019**

**Instructions:**
*Turn off and put away your cell phone.*
*Please write your Name and Lehman ID # on the top of this page.*
*Please sign and date the pledge below to comply with the Code of Academic Integrity.*
*No consultation material, calculators, or electronic devices are allowed during the exam.*
*If any question is unclear, raise your hand to ask for clarifications.*
*The regular amount of time you have to complete the exam is 100 minutes.*
**You must show all of your work!** *No credit will be given for unsupported answers.*
*Please try to be as organized, objective, and logical as possible in your answers.*

| # | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 15 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 15 | |
| Total | 100 | |

My signature below certifies that I complied with the CUNY Academic Integrity Policy and the Lehman College Code of Academic Integrity in completing this examination.

Signature ____________________ Date ____________

**Problem 1 (10 pts):** Compute the following limits, or explain why they do not exist:

a) **(5 pts)** $\lim_{x\to 3} \frac{x^2-5x+6}{x-3}$

$$\lim_{x\to 3} \frac{x^2-5x+6}{x-3} = \lim_{x\to 3} \frac{(x-3)(x-2)}{x-3} =$$

$$= \lim_{x\to 3} x-2 = \boxed{1}\,//$$

b) **(5 pts)** $\lim_{x\to 0} \frac{2\sin(4x)\cos(7x)}{x}$

$$\lim_{x\to 0} \frac{2\sin(4x)\cos(7x)}{x} = \lim_{x\to 0} 2\cos(7x) \cdot \overbrace{\left(\frac{\sin(4x)}{4x}\right)}^{1} \cdot 4$$

$$= 2 \cdot 1 \cdot 4 = \boxed{8}\,//$$

**Problem 2 (15 pts):** What is the equation of the tangent line to the graph of the function $f(x) = x^4 + 3\sin(x) + e^x$ at the point $(0, 1)$?

Slope of tangent line is $m = f'(0)$:

$f'(x) = 4x^3 + 3\cos x + e^x$

$f'(0) = 3 + e^0 = 4$

So $\boxed{m = 4}$

Equation of the line: $y = mx + b$

$y = 4x + b$ passes through $(0,1)$, so:

$1 = 4 \cdot 0 + b \Rightarrow \boxed{b = 1}$

Thus, the equation of the tangent line we are looking for is:

$$\boxed{y = 4x + 1}$$

**Problem 3 (10 pts):** For what value of $a$ is the function $f(x)$ below continuous everywhere? Justify your answer.

$$f(x) = \begin{cases} ax^2 - 1 & \text{if } x \le 1 \\ \dfrac{\sqrt{x+3}-2}{x-1} & \text{if } x > 1 \end{cases}$$

The function $f(x)$ is clearly continuous at all $x \neq 1$, since it is a composition of continuous functions. For continuity at $x = 1$, we need the following lateral limits to agree:

- $\lim_{x \to 1_-} f(x) = \lim_{x \to 1_-} ax^2 - 1 = \boxed{a - 1}$

- $\lim_{x \to 1_+} f(x) = \lim_{x \to 1_+} \dfrac{\sqrt{x+3}-2}{x-1} = \lim_{x \to 1_+} \dfrac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(\sqrt{x+3}+2)}$

$$= \lim_{x \to 1_+} \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} = \lim_{x \to 1_+} \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x+3}+2)} =$$

$$= \lim_{x \to 1_+} \frac{1}{\sqrt{x+3}+2} = \boxed{\frac{1}{4}}$$

Thus, we need $a - 1 = \frac{1}{4}$, so $\boxed{a = \frac{5}{4}}$

**Problem 4 (10 pts):** Compute the following limits, or explain why they do not exist:

a) **(5 pts)** $\lim_{x \to +\infty} \frac{x^5 - 3x^4 - 2}{x^6 - 2x + 1}$

$$\lim_{x \to +\infty} \frac{x^5 - 3x^4 - 2}{x^6 - 2x + 1} = \lim_{x \to +\infty} \frac{x^5\left(1 - \frac{3}{x} - \frac{2}{x^5}\right)}{x^6\left(1 - \frac{2}{x^5} + \frac{1}{x^6}\right)}$$

$$= \lim_{x \to +\infty} \frac{1 - \frac{3}{x} \to 0 - \frac{2}{x^5} \to 0}{\underbrace{x}_{+\infty}\left(1 - \underbrace{\frac{2}{x^5}}_{0} + \underbrace{\frac{1}{x^6}}_{0}\right)} = \boxed{0.}$$

b) **(5 pts)** $\lim_{t \to 2} \frac{16}{t-2}$

$$\lim_{t \to 2^-} \frac{16}{t-2} = -\infty \quad \text{and} \quad \lim_{x \to 2^+} \frac{16}{t-2} = +\infty$$

So the limit $\boxed{\text{does not exist!}}$

**Problem 5 (10 pts):** Show that the derivative of $f(x) = x^2 + 1$ is equal to $f'(x) = 2x$ **using the definition as a limit.**

According to the definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2+1)}{h} =$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} =$$

$$= \lim_{h \to 0} 2x + h = \boxed{2x}$$

**Problem 6 (10 pts):** Compute the first derivative of the following functions:

a) **(5 pts)** $s(t) = \dfrac{t^4+2}{e^t\cos(t)}$

$$s'(t) = \frac{4t^3 e^t \cos t - (t^4+2)(e^t \cos t + e^t(-\sin t))}{e^{2t}\cos^2 t}$$

b) **(5 pts)** $A(x) = \left(x^2\ln(x)+3\right)^7$

$$A'(x) = 7(x^2\ln x + 3)^6 \cdot \left(2x\ln x + x^2 \cdot \frac{1}{x}\right)$$

$$= 7(x^2\ln x+3)^6(2x\ln x + x)$$

**Problem 7 (10 pts):** Find the slope of the tangent line to the curve defined by the equation $2(x^2 + y^2)^3 = xy + 15$ at the point $(1, 1)$.

By implicit differentiation with respect to x, we have:

$$2 \cdot 3(x^2+y^2)^2 \cdot \left(2x + 2y \frac{dy}{dx}\right) = y + x \frac{dy}{dx}$$

$$12(x^2+y^2)^2 \cdot \left(x + y \frac{dy}{dx}\right) = y + x \frac{dy}{dx}$$

Setting $x = 1$ and $y = 1$, we have,

$$12 \cdot 2^2 \left(1 + \frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$$

$$48 + 48 \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$47 \frac{dy}{dx} = -47$$

$$\boxed{\frac{dy}{dx} = -1}$$

So the slope of the tangent line to the given curve at $(1,1)$ is $m = -1$.

**Problem 8 (10 pts):** Let $g(x)$ be the inverse function of $f(x)$, so that $f(g(x)) = x$ and $g(f(x)) = x$. Use the following table to compute the quantities asked below.

$$\begin{array}{lll} f(0) = 1 & f(1) = 0 & f(2) = 3 \\ f'(0) = 6 & f'(1) = 2 & f'(2) = 5 \\ f''(0) = 3 & f''(1) = 8 & f''(2) = 0 \end{array}$$

a) **(5 pts)** $g'(0) =$

$f(g(x)) = x \implies f'(g(x))\,g'(x) = 1$

$\implies f'(g(0))\,g'(0) = 1.$

$\implies g'(0) = \dfrac{1}{f'(g(0))}$

Since $f(1) = 0$, $g(0) = 1$. Thus,

$g'(0) = \dfrac{1}{f'(1)} = \boxed{\dfrac{1}{2}}$

b) **(5 pts)** $g''(0) =$

$f'(g(x))\,g'(x) = 1 \implies f''(g(x))\,g'(x)^2 + f'(g(x))\,g''(x) = 0$

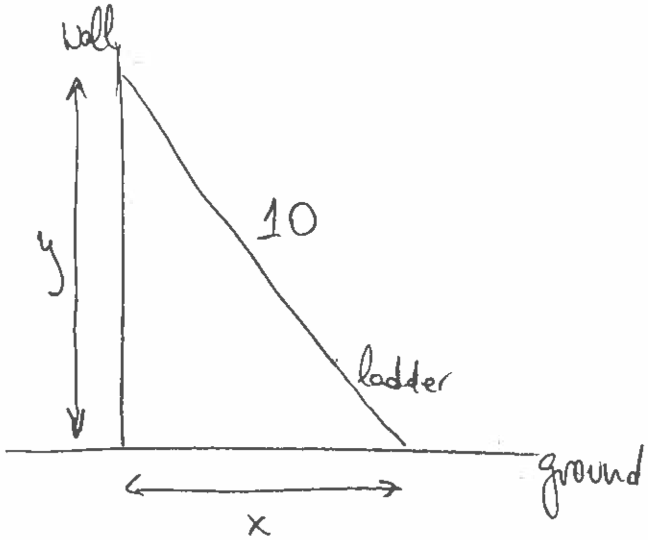
$\implies f''(g(0))\,g'(0)^2 + f'(g(0))\,g''(0) = 0$

$\implies f''(1)\left(\frac{1}{2}\right)^2 + f'(1)\,g''(0) = 0$

$\implies 8 \cdot \frac{1}{4} + 2\,g''(0) = 0$

$\implies 2 + 2g''(0) = 0 \implies \boxed{g''(0) = -1}$

**Problem 9 (15 pts):** A ladder 10 ft long is resting against a vertical wall. The bottom of the ladder is sliding away from the wall at a speed of 1 ft per second. When the base of the ladder is 6 ft away from the wall, how fast is the top sliding down the wall?




$x$ = distance from bottom of the ladder to wall

$y$ = height of the top of the ladder.

$$x^2 + y^2 = 10^2$$

We have: $\frac{dx}{dt} = 1$ ft/s. We want $\frac{dy}{dt}$ when $x = 6$.

Differentiating $x^2 + y^2 = 10^2$ with respect to time:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

When $x = 6$, $6^2 + y^2 = 10^2$ so $\underline{y = 8}$. Thus, we have

$$6 \cdot 1 + 8 \cdot \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{6}{8} = -\frac{3}{4}$$

So the top is sliding down at a $\boxed{\text{speed of } 3/4 \text{ ft/s}}$