MAT 17C Solutions to HW9

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{3^n}$ converges by the Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+2)^2}{3^{n+1}} \cdot \frac{3^n}{(n+1)^2} = \lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^2 \frac{1}{3} = \frac{1}{3} < 1$$

b) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n!}$ converges absolutely by the Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

c) $\displaystyle\sum_{n=1}^{+\infty} \frac{5n^2}{(2n)!}$ converges by the Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{5(n+1)^2}{(2n+2)!}\,\frac{(2n)!}{5n^2} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 \frac{1}{(2n+2)(2n+1)}$$

$$= 0 < 1$$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^{5/2}}{n^3+2}$ converges conditionally.

Alternating Series Test: $\dfrac{n^{5/2}}{n^3+2}$ is positive, decreasing

and $\displaystyle\lim_{n\to\infty} \frac{n^{5/2}}{n^3+2} = 0$ so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^{5/2}}{n^3+2} < \infty$.

However, by Limit Comparison test with $b_n = \frac{1}{\sqrt{n}}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{5/2}\cdot n^{1/2}}{n^3+2} = \lim_{n\to\infty} \frac{n^3}{n^3+2} = 1.$$

So $\sum_{n=1}^{\infty} \frac{n^{5/2}}{n^3+2}$ diverges because $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

(p-series, $p = 1/2$).

e) $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)!}$ Converges

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n!}{(n+2)!} \frac{(n+1)!}{(n-1)!} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}\ n\cancel{(n-1)!}}{(n+2)\cancel{(n+1)!}\ \cancel{(n-1)!}}$$

$$= \lim_{n\to\infty} \frac{n}{n+2} = 1 \quad \text{inconclusive!}$$

$$\frac{(n-1)!}{(n+1)!} = \frac{1}{(n+1)\,n} = \frac{1}{n^2+n} =: a_n$$

Limit comp. test with $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = 1.$$

So $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)!}$ converges because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

(p-series, $p=2$).