SOLUTION TO PRACTICE PROBLEMS FOR MIDTERM EXAM

1 a) $$\int x \sin x = -x\cos x + \int \cos x \, dx = \boxed{-x\cos x + \sin x + C}$$

b) $$\int x e^{-x} dx = -xe^{-x} + \int e^{-x} dx = \boxed{-xe^{-x} - e^{-x} + C}$$

c) $$\int x^2 \ln x \, dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x} dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\frac{x^3}{3} + C = \boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + C}$$

d) $$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$

$$= x^2 e^x - 2\left(xe^x - \int e^x dx\right)$$

$$\boxed{= x^2 e^x - 2xe^x + 2e^x + C}$$

e) $$\int x\sqrt{9-x^2}\, dx = \int \sqrt{u}\left(-\frac{du}{2}\right) = -\frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$u = 9 - x^2$

$du = -2x dx$

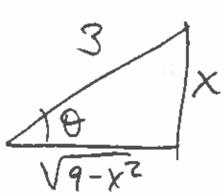
$$\boxed{= -\frac{1}{3}(9-x^2)^{3/2} + C}$$

f) $$\int \sqrt{9-x^2}\, dx = \int \sqrt{9(1-\sin^2\theta)}\; 3\cos\theta \, d\theta = 9\int \cos^2\theta \, d\theta$$

$x = 3\sin\theta$

$dx = 3\cos\theta \, d\theta$

$$= 9\int \frac{1+\cos 2\theta}{2} d\theta = \frac{9}{2}\left(\theta + \frac{\overbrace{\sin 2\theta}^{2\sin\theta\cos\theta}}{2}\right) + C$$



$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$$

$$\boxed{= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C}$$

2) a) $\int_0^{+\infty} x \sin x \, dx = \lim_{b\to\infty} \int_0^b x \sin x \, dx =$ ← from 1 a)

$= \lim_{b\to\infty} -b\cos b + \sin b - 0$

This limit does not exist, so the improper integral does not converge.

b) $\int_0^{+\infty} x e^{-x} dx = \lim_{b\to\infty} \int_0^b x e^{-x} dx =$ ← from 2 a)

$= \lim_{b\to\infty} \underset{\downarrow \atop 0}{-be^{-b}} - \underset{\downarrow \atop 0}{e^{b}} - (-e^0) = e^0 = \boxed{1}$

by L'Hospital

c) $\int_1^{+\infty} \frac{3}{x^2} dx = \lim_{b\to\infty} \int_1^b 3x^{-2} dx = \lim_{b\to\infty} \frac{3x^{-1}}{-1}\Big|_1^b =$

$= \lim_{b\to\infty} -\frac{3}{b} + 3 = \boxed{3}$

d) $\int_0^1 \frac{3}{x^2} dx = \lim_{a\to 0_+} \int_a^1 3x^{-2} dx = \lim_{a\to 0_+} \frac{3x^{-1}}{-1}\Big|_a^1 =$

$= \lim_{a\to 0_+} -3 + \frac{3}{a} = \boxed{+\infty}$ Does not converge

e) $\int_{-\infty}^{+\infty} \frac{e^x}{e^x+1} dx = \lim_{a\to-\infty} \int_a^0 \frac{e^x dx}{e^x+1} + \lim_{b\to+\infty} \int_0^b \frac{e^x dx}{e^x+1} =$

$= \lim_{a\to-\infty} \ln(e^x+1)\Big|_a^0 + \lim_{b\to+\infty} \ln(e^x+1)\Big|_0^b =$

$= \lim_{a\to-\infty} \ln 2 - \underset{\downarrow \atop \infty}{\ln(e^a+1)} + \lim_{b\to\infty} \underset{\downarrow \atop \infty}{\ln(e^b+1)} - \ln(2)$

Does not converge.

f) $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{-2x}}{4\pi}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{e^{-2x}}{4\pi}\,dx + \lim_{b\to+\infty}\int_0^b \frac{e^{-2x}}{4\pi}\,dx =$

$$= \lim_{a\to-\infty} \frac{1}{4\pi}\left(\frac{e^{-2x}}{-2}\right)\Bigg|_a^0 + \lim_{b\to+\infty} \frac{1}{4\pi}\left(\frac{e^{-2x}}{-2}\right)\Bigg|_0^b =$$

$$= \lim_{a\to-\infty} -\frac{1}{8\pi}\Big(1-\underbrace{e^{-2a}}_{\downarrow\ \infty}\Big) + \lim_{b\to+\infty} -\frac{1}{8\pi}\Big(\underbrace{e^{-2b}}_{\downarrow\ 0}-1\Big) = +\infty$$

Does not converge

3) a) $\displaystyle\int \frac{3}{\sqrt{9-x^2}}\,dx = \int \frac{9\cancel{\cos\theta}\,d\theta}{3\cancel{\cos\theta}} = \int 3\,d\theta = 3\theta + C$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$$= \boxed{3\arcsin\left(\frac{x}{3}\right) + C}$$

b) $\displaystyle\int \sqrt{1+t^2}\,dt = \int \sec\theta \cdot \sec^2\theta\,d\theta = \int \sec^3\theta\,d\theta =$

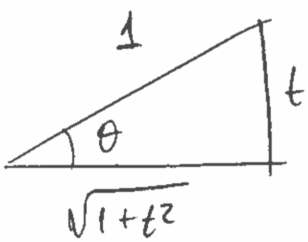
$t = \tan\theta$

$dt = \sec^2\theta\,d\theta$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

Doing parts, moving $\int \sec^3\theta\,d\theta$ back to other side...

$$= \frac{1}{2}\,\frac{1}{\sqrt{1+t^2}}\cdot t + \frac{1}{2}\ln\left|\frac{1}{\sqrt{1+t^2}} + t\right| + C$$



$$= \boxed{\frac{t}{2\sqrt{1+t^2}} + \ln\sqrt{t + \frac{1}{\sqrt{1+t^2}}} + C}$$

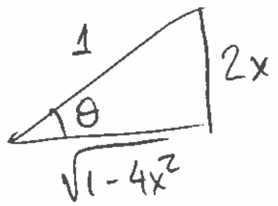
c) $\displaystyle\int \sqrt{1-4x^2}\,dx = \int \cos\theta \, \frac{\cos\theta\, d\theta}{2} = \frac{1}{2}\int \frac{1+\cos 2\theta}{2}\, d\theta$

$2x = \sin\theta$

$2\,dx = \cos\theta\, d\theta$

$2\sin\theta\cos\theta$

$$= \frac{1}{2}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C = \frac{\theta}{4} + \frac{\sin 2\theta}{8} + C$$



$$= \frac{1}{4}\arcsin(2x) + \frac{1}{4}\cdot 2x \cdot \sqrt{1-4x^2} + C$$

$$\boxed{= \frac{1}{4}\arcsin(2x) + \frac{x\sqrt{1-4x^2}}{2} + C}$$

4) a) $\displaystyle\frac{dy}{dx} = \sin^5 x \cos^6 x \implies y = \int \sin^5 x \cos^6 x\, dx =$

$$= \int (1-\cos^2 x)^2 \cos^6 x \cdot \sin x\, dx = -\int (1-u^2)^2 u^6\, du$$

$u = \cos x$

$du = -\sin x\, dx$

$$= -\int (1-2u^2+u^4)u^6\, du = -\int u^6 - 2u^8 + u^{10}\, du =$$

$$= -\frac{u^7}{7} + 2\frac{u^9}{9} - \frac{u^{11}}{11} + C = -\frac{1}{7}\cos^7 x + \frac{2}{9}\cos^9 x - \frac{1}{11}\cos^{11} x + C$$

$$y(0) = 0 \implies -\frac{1}{7} + \frac{2}{9} - \frac{1}{11} + C = 0 \implies C = \frac{8}{693}$$

$$\boxed{y(x) = -\frac{1}{7}\cos^7 x + \frac{2}{9}\cos^9 x - \frac{1}{11}\cos^{11} x + \frac{8}{693}}$$

b) $\frac{dy}{dx} = x \sin^2 x \Rightarrow y = \int x \sin^2 x \, dx = \dots$

(Integrate by parts, using that

$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4}$)

$$\dots = x \cdot \left(\frac{x}{2} - \frac{\sin 2x}{4} \right) - \int \left(\frac{x}{2} - \frac{\sin 2x}{4} \right) dx$$

$$= \frac{x^2}{2} - \frac{x \sin 2x}{4} - \frac{x^2}{4} - \frac{1}{4} \cdot \frac{\cos 2x}{2} + c, \quad y(0) = 0 \Rightarrow c = 0.$$

$$\boxed{y(x) = \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8}}$$

c) $\frac{dy}{dx} = \sqrt{4-x^2} \Rightarrow y = \int \sqrt{4-x^2} \, dx \overset{\substack{x = 2\sin\theta \\ dx = 2\cos\theta \, d\theta}}{=} \int 2 \cdot \cos\theta \cdot 2\cos\theta \, d\theta =$

$$= 4 \int \cos^2\theta \, d\theta = 4 \int \frac{1+\cos 2\theta}{2} d\theta = 2\left(\theta + \frac{\sin 2\theta}{2}\right) + C =$$

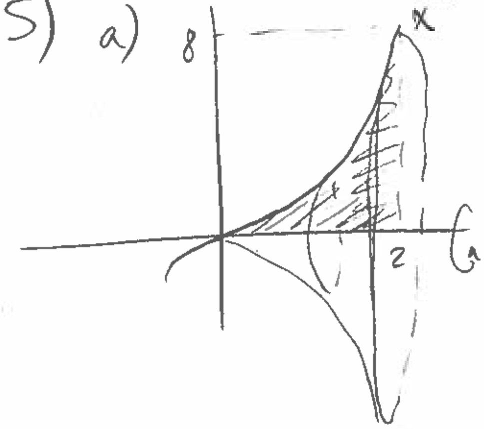
$$= 2 \arcsin\left(\frac{x}{2}\right) + 2 \cdot \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2} + C$$

$$= 2 \arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C$$

$y(0) = 1 \Rightarrow C = 1$

$$\boxed{y(x) = 2 \arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + 1}$$

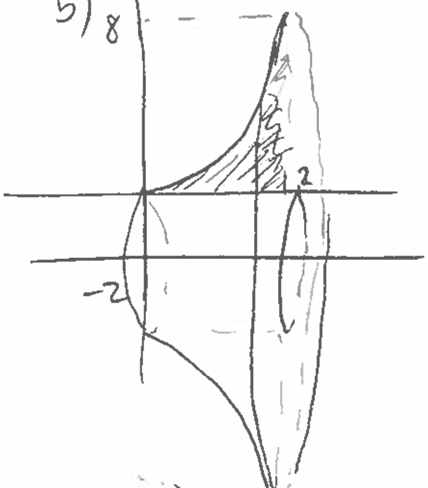
5) a)



Disk Method:

$$V = \pi \int_0^2 (x^3)^2 \, dx = \pi \int_0^2 x^6 \, dx$$

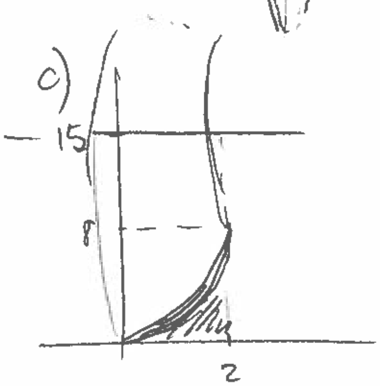
b)



Washer method:

$$V = \pi \int_0^2 (x^3+2)^2 - 2^2 \, dx$$

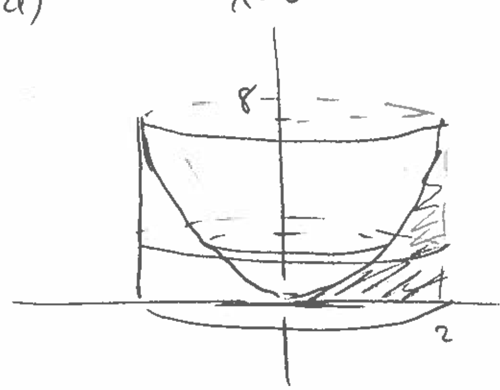
c)



Washer method

$$V = \pi \int_0^2 15^2 - (15 - x^3)^2 \, dx$$

d)



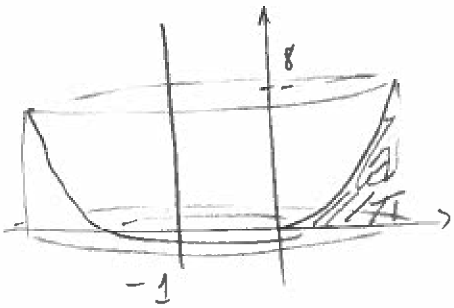
Shell Method:

$$V = 2\pi \int_0^2 x \cdot x^3 \, dx = 2\pi \int_0^2 x^4 \, dx$$

shell radius

shell height

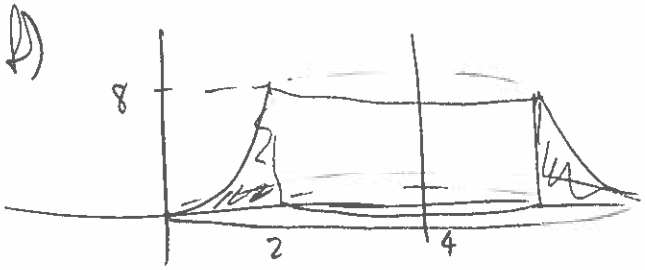
e)



Shell Method

$$V = 2\pi \int_0^2 \underbrace{(x+1)}_{\text{shell radius}} \underbrace{x^3}_{\text{shell height}} \, dx$$

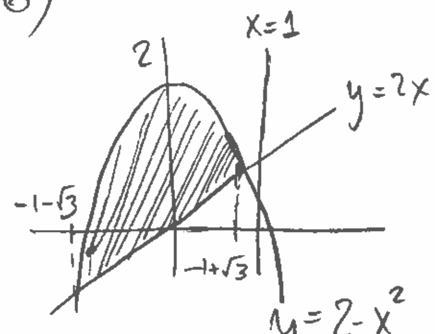
f)



Shell Method

$$V = 2\pi \int_0^2 \underbrace{(4-x)}_{\text{shell radius}} \underbrace{x^3}_{\text{shell height}} \, dx$$

6)



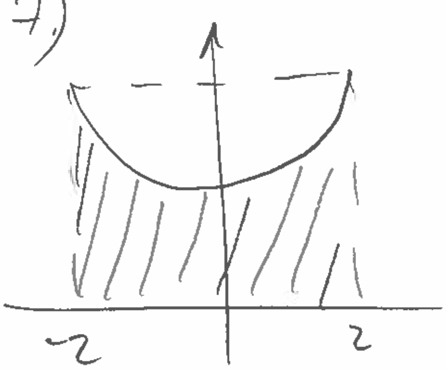
$2x = 2 - x^2 \iff x = -1 \pm \sqrt{3} < 1$

Shell Method:

$$V = 2\pi \int_{-1-\sqrt{3}}^{-1+\sqrt{3}} \underbrace{(1-x)}_{\text{shell radius}} \underbrace{(2-x^2-2x)}_{\text{shell height}} \, dx =$$

$$= 2\pi \int_{-1-\sqrt{3}}^{-1+\sqrt{3}} x^3 + x^2 - 4x + 2 \, dx = \dots = \boxed{16\pi\sqrt{3}}$$

7)



Disk Method:

$$V = \pi \int_{-2}^{2} \left(\frac{e^x + e^{-x}}{2}\right)^2 dx = \pi \int_{-2}^{2} \frac{e^{2x} + 2 + e^{-2x}}{4} \, dx$$

$$= \frac{\pi}{4} \frac{e^{2x}}{2} \Big|_{-2}^{2} + \frac{\pi}{2} \cdot 4 + \frac{\pi}{4} \frac{e^{-2x}}{-2} \Big|_{-2}^{2} = \boxed{\frac{\pi}{4}\left(e^4 - e^{-4}\right) + 2\pi}$$

8) a) 20% of 300 mg $= \frac{2}{10} \cdot 300 = \boxed{60\text{ mg}}$

b) From 1^{st} day: 20% of 20% of 300mg: $\frac{2}{10} \cdot \frac{2}{10} \cdot 300$

From 2^{nd} day: 20% of 300 mg $= \frac{2}{10} \cdot 300$

Total $= \frac{2}{10} \cdot \frac{2}{10} \cdot 300 + \frac{2}{10} \cdot 300 = 12 + 60 = \boxed{72\text{ mg}}$

c) From 1^{st} day: 20% of 20% of 20% of 300mg: $\left(\frac{2}{10}\right)^3 \cdot 300$

From 2^{nd} day: 20% of 20% of 300 mg: $\left(\frac{2}{10}\right)^2 \cdot 300$

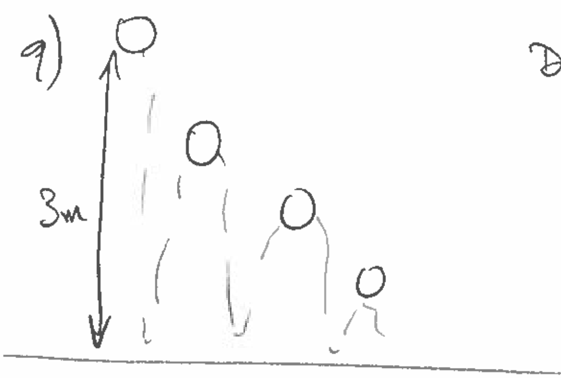
From 3^{rd} day: 20% of 300 mg: $\frac{2}{10} \cdot 300$

total $= \left(\frac{2}{10}\right)^3 \cdot 300 + \left(\frac{2}{10}\right)^2 \cdot 300 + \frac{2}{10} \cdot 300$

$= 2.4 + 72 = \boxed{74.4\text{ mg}}$

d)
$$\sum_{n=1}^{+\infty} \left(\frac{2}{10}\right)^n \cdot 300 = 300 \cdot \frac{2/10}{1 - 2/10} = \frac{300}{5} \cdot \frac{1}{4/5}$$

$$= \frac{300}{4} = \boxed{75\text{ mg}}$$

9) 

$$\overset{\text{Drop}}{3} + \overbrace{\underset{\text{up \& down}}{2} \cdot \underbrace{\frac{2}{3} \cdot 3}_{\text{2/3 of previous height}}}^{1^{st}\text{ bounce}} + \overbrace{\underset{\text{up \& down}}{2} \cdot \underbrace{\left(\frac{2}{3}\right)^2 \cdot 3}_{\text{2/3 of previous height}}}^{2^{nd}\text{ bounce}} + \dots =$$

$$= 3 + \sum_{n=1}^{+\infty} 2 \left(\frac{2}{3}\right)^n 3 = 3 + 6 \frac{2/3}{1 - 2/3}$$

$$= 3 + 6 \frac{2/3}{1/3} = 3 + 12 = \boxed{15\text{ m}}$$

10) a) $\sum_{n=1}^{+\infty} n^2 = +\infty$ diverges by n^{th} term test.

$$\lim_{n\to\infty} n^2 = +\infty \neq 0.$$

b) $\sum_{n=1}^{+\infty} \frac{2}{5^n} = \frac{2/5}{1-1/5} = \frac{2}{4} = \frac{1}{2} < \infty$ converges, b/c it is Geometric w/ $r = \frac{1}{5} < 1$

c) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1} = +\infty$ diverges by n^{th} term test.

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1 \neq 0.$$

d) $\sum_{n=1}^{\infty} \frac{1}{n^2+4} < \infty$ converges By Direct Comparison w/ p-series, $p=2$;

$$\forall n, \quad \frac{1}{n^2+4} < \frac{1}{n^2} \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+4} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

(p-series) $p=2>1$

e) $\sum_{n=1}^{\infty} \frac{n}{n^2+4} = +\infty$ diverges by Limit Comp. Test w/ harmonic series (p-series, $p=1$);

$$a_n = \frac{n}{n^2+4}, \quad b_n = \frac{1}{n} \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+4} = 1 = L$$

$L < +\infty$, $L > 0$

$$\text{so } \sum_{n=1}^{\infty} \frac{1}{n} = +\infty \Longrightarrow \sum_{n=1}^{\infty} \frac{n}{n^2+4} = +\infty \text{ as well.}$$

f) $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}+1} = +\infty$ diverges by Limit Comp. Test w/ p-series, $p = 1/2 < 1$;

$$a_n = \frac{1}{\sqrt{n}+1}, \quad b_n = \frac{1}{\sqrt{n}}, \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = 1 = L$$

$L < +\infty$, $L > 0$

$$\text{So } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = +\infty \Longrightarrow \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}+1} = +\infty \text{ as well.}$$

(p-series, $p = 1/2 < 1$)

g) $\sum_{n=1}^{\infty} \frac{n}{e^n} < \infty$ converges by Integral Test, $a_n = f(n)$ where

$f(x) = \frac{x}{e^x}$ is positive, continuous and decreasing.

From exercise 2 b), we have

$$\int_1^{+\infty} \frac{x}{e^x}\,dx < \int_0^{+\infty} \frac{x}{e^x}\,dx = 1.$$

Thus, $\sum_{n=1}^{\infty} \frac{n}{e^n} < \infty$ converges.