MAT226

Solutions to HW 7

1. $f(x,y) = x^3 + y^3 - 3xy + 4$

$$\nabla f(x,y) = \left(3x^2 - 3y,\ 3y^2 - 3x\right)$$

Critical points:

$$\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \iff \begin{cases} x^2 = y \\ y^2 = x \end{cases}$$

Substituting: $(x^2)^2 = x \Rightarrow x^4 = x \Rightarrow x(x^3 - 1) = 0$

$$\Rightarrow \boxed{x = 0} \text{ or } \boxed{x = 1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\boxed{y = x^2 = 0} \qquad \boxed{y = x^2 = 1}$$

So there are 2 critical points: $(0,0)$ and $(1,1)$.

$$(\text{Hess } f)(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

$$(\text{Hess } f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} \Rightarrow \det (\text{Hess } f)(0,0) = -9 < 0$$

$\Rightarrow$ $\boxed{(0,0) \text{ is a saddle point.}}$

$$(\text{Hess } f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \Rightarrow \det (\text{Hess } f)(1,1) = 36 - 9 > 0$$

$$\frac{\partial^2 f}{\partial x^2}(1,1) = 6 > 0$$

$\Rightarrow$ $\boxed{(1,1) \text{ is a local minimum.}}$

2. $f(x,y) = x^3 - x^2 - y^2 + 3xy^2 + 1$

$\nabla f(x,y) = \left(3x^2 - 2x + 3y^2,\ -2y + 6xy\right)$

Crit. pts: $\begin{cases} 3x^2 - 2x + 3y^2 = 0 \\ -2y + 6xy = 0 \rightsquigarrow y(6x-2) = 0 \end{cases}$

$\Downarrow$

$\boxed{y = 0 \quad \text{or} \quad x = \frac{1}{3}}$

$\boxed{\text{If } y=0}$: $3x^2 - 2x = 0$

$x(3x-2) = 0 \Rightarrow x = 0 \quad \text{or} \quad x = \frac{2}{3}$

$\boxed{\text{If } x = \frac{1}{3}}$: $\frac{1}{3} - \frac{2}{3} + 3y^2 = 0 \Rightarrow 3y^2 = \frac{1}{3}$

$\Rightarrow y^2 = \frac{1}{9} \Rightarrow \boxed{y = \pm\frac{1}{3}}$

there are 4 critical points: $(0,0)$, $\left(\frac{2}{3}, 0\right)$, $\left(\frac{1}{3}, \frac{1}{3}\right)$, $\left(\frac{1}{3}, -\frac{1}{3}\right)$

$$(\text{Hess } f)(x,y) = \begin{pmatrix} 6x-2 & 6y \\ 6y & 6x-2 \end{pmatrix}$$

$$(\text{Hess } f)(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \begin{array}{l} \det(\text{Hess } f)(0,0) = 4 > 0 \\ \frac{\partial^2 f}{\partial x^2}(0,0) = -2 < 0 \end{array}$$

$\Rightarrow \boxed{(0,0) \text{ is a local maximum}}$

$$(\mathrm{Hess}\, f)\left(\frac{2}{3},0\right) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \begin{array}{l} \det(\mathrm{Hess}\, f)\left(\frac{2}{3},0\right) = 4 > 0 \\ \frac{\partial^2 f}{\partial x^2}\left(\frac{2}{3},0\right) = 2 > 0 \end{array}$$

$$\Rightarrow \boxed{\left(\frac{2}{3},0\right) \text{ is a } \underline{\text{local minimum}}}$$

$$(\mathrm{Hess}\, f)\left(\frac{1}{3},\frac{1}{3}\right) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \Rightarrow \det(\mathrm{Hess}\, f)\left(\frac{1}{3},\frac{1}{3}\right) = -4 < 0$$

$$\Rightarrow \boxed{\left(\frac{1}{3},\frac{1}{3}\right) \text{ is a } \underline{\text{saddle pt.}}}$$

$$(\mathrm{Hess}\, f)\left(\frac{1}{3},-\frac{1}{3}\right) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \Rightarrow \det(\mathrm{Hess}\, f)\left(\frac{1}{3},-\frac{1}{3}\right) = -4 < 0$$

$$\Rightarrow \boxed{\left(\frac{1}{3},-\frac{1}{3}\right) \text{ is a } \underline{\text{saddle pt.}}}$$