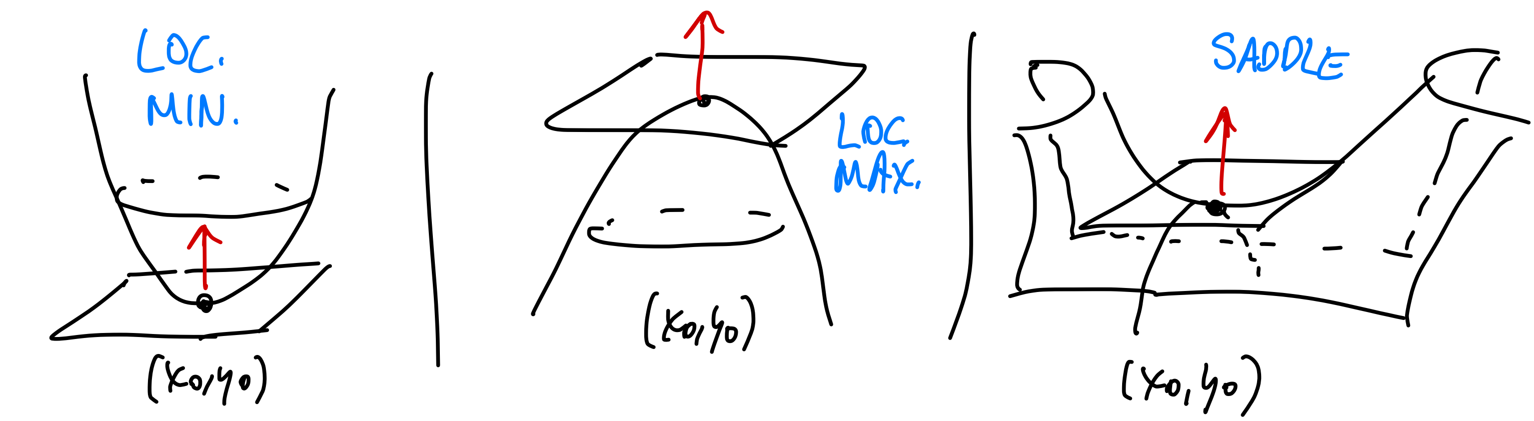
Relative extrema: $z = f(x,y)$



If (x_0,y_0) is a critical point of $f(x,y)$ (i.e. $\nabla f(x_0,y_0)=0$); then, to decide if it is a local min, or local max, or saddle use:

$$(\text{Hess } f)(x_0,y_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x_0,y_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) \\ \frac{\partial^2 f}{\partial y \partial x}(x_0,y_0) & \frac{\partial^2 f}{\partial y^2}(x_0,y_0) \end{bmatrix}$$

Quick reminder: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$$\det(\text{Hess } f) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

Second Derivative Test:

1. If $\det(\text{Hess } f) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, then the point is a <u>local minimum</u>.
2. If $\det(\text{Hess } f) > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, then the point is a <u>local maximum</u>.
3. If $\det(\text{Hess } f) < 0$, then the point is a <u>saddle</u>.

<u>Note</u>: If $\det(\text{Hess } f) = 0$, then test is <u>inconclusive</u>.

FOR THOSE OF YOU THAT KNOW LINEAR ALGEBRA:

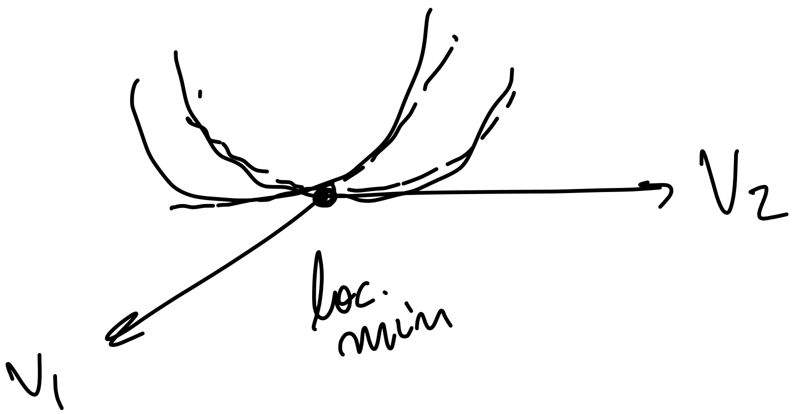
In terms of <u>eigenvalues</u> λ_1, λ_2 of Hess f

If $\det(\text{Hess } f) = \lambda_1 \cdot \lambda_2 > 0$ and $\lambda_1 > 0$ then point is a <u>local min</u>.

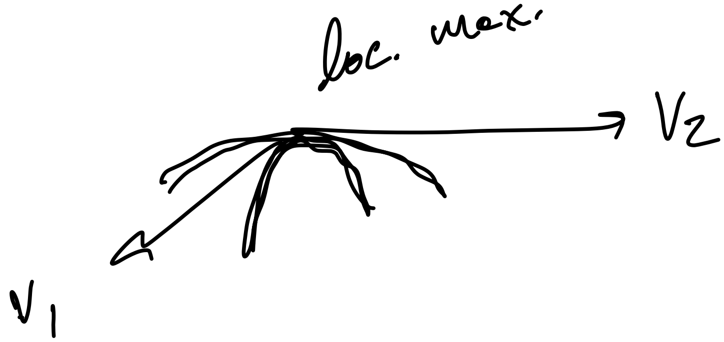
If $\det(\text{Hess } f) = \lambda_1 \cdot \lambda_2 > 0$ and $\lambda_1 < 0$ then point is a <u>local max</u>.

If $\det(\text{Hess } f) = \lambda_1 \cdot \lambda_2 < 0$, then point is a <u>saddle</u>.

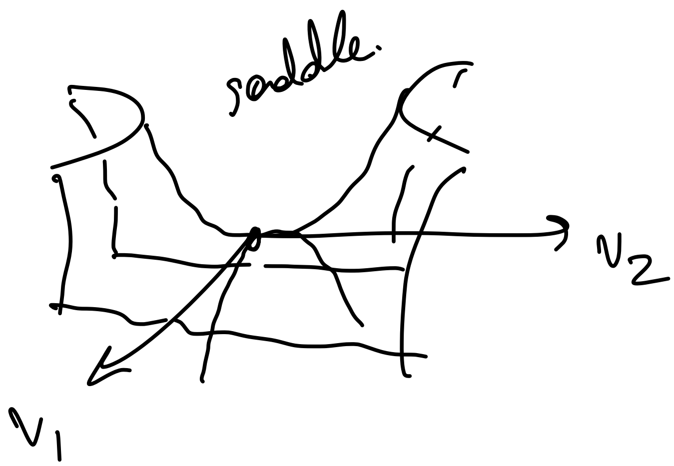
$\lambda_1 > 0$
$\lambda_2 > 0$



$\lambda_1 < 0$
$\lambda_2 < 0$



$\lambda_1 < 0$
$\lambda_2 > 0$



Example: Find all critical points of

$$f(x,y) = x^3 - x^2 - y^2 + 3xy^2 + 1$$

and classify them into loc. min, loc. max, saddle.

From previous video:

Second Derivative Test:

1. If $\det(\text{Hess } f) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, then the point is a local minimum.
2. If $\det(\text{Hess } f) > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, then the point is a local maximum.
3. If $\det(\text{Hess } f) < 0$, then the point is a saddle.

Critical points:

$$\nabla f(x,y) = (\underbrace{3x^2 - 2x + 3y^2}_{=\frac{\partial f}{\partial x}}, \underbrace{-2y + 6xy}_{=\frac{\partial f}{\partial y}}) = (0,0)$$

$$\begin{cases} 3x^2 + 3y^2 - 2x = 0 \\ 3xy - y = 0 \implies y(3x-1) = 0 \implies \begin{cases} y = 0 \checkmark \\ \text{or} \\ x = 1/3 \end{cases} \end{cases}$$

If $y = 0$:
$$3x^2 + 3 \cdot 0^2 - 2x = 0$$
$$3x^2 - 2x = 0$$
$$x(3x-2) = 0 \implies x = 0 \text{ or } x = 2/3$$

If $x = 1/3$:
$$\frac{1}{3} + 3y^2 - \frac{2}{3} = 0$$
$$3y^2 - \frac{1}{3} = 0 \implies 3y^2 = \frac{1}{3} \implies y^2 = \frac{1}{9}$$
$$\implies y = \pm\sqrt{\frac{1}{9}} = \pm\frac{1}{3}$$

$$\boxed{\text{Critical points: } (0,0), \left(\frac{2}{3}, 0\right), \left(\frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, -\frac{1}{3}\right).}$$

$$\text{Hess } f(x,y) = \begin{bmatrix} 6x-2 & 6y \\ 6y & -2+6x \end{bmatrix}$$

① $\text{Hess } f(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

- $\det(\text{Hess } f)(0,0) = 4 > 0$
- $\dfrac{\partial^2 f}{\partial x^2}(0,0) = -2 < 0$

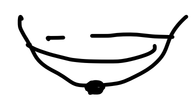
$\Longrightarrow$ $\boxed{(0,0) \text{ is a local max.}}$


$(0,0)$

② $(\text{Hess } f)\left(\frac{2}{3}, 0\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

- $\det(\text{Hess } f)\left(\frac{2}{3}, 0\right) = 4 > 0$
- $\dfrac{\partial^2 f}{\partial x^2}\left(\frac{2}{3}, 0\right) = 2 > 0$

$\Longrightarrow$ $\boxed{\left(\frac{2}{3}, 0\right) \text{ is a local min.}}$


$\left(\frac{2}{3}, 0\right)$

③ $(\text{Hess } f)\left(\frac{1}{3}, \frac{1}{3}\right) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

- $\det(\text{Hess } f)\left(\frac{1}{3}, \frac{1}{3}\right) = -4 < 0$

$\Longrightarrow$ $\boxed{\left(\frac{1}{3}, \frac{1}{3}\right) \text{ is a saddle.}}$

④ $(\text{Hess } f)\left(\frac{1}{3}, -\frac{1}{3}\right) = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$

- $\det(\text{Hess } f)\left(\frac{1}{3}, -\frac{1}{3}\right) = -4 < 0$

$\Longrightarrow$ $\boxed{\left(\frac{1}{3}, -\frac{1}{3}\right) \text{ is a saddle}}$