MAT 226

Lecture 16

March 29 - 30 (online!)

Review Problems for the Midterm Exam

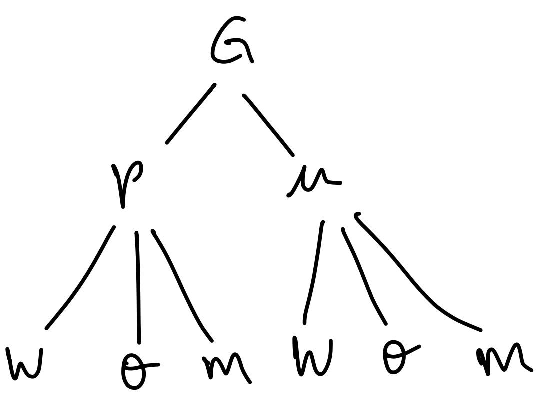
#1

Problem 5 (9 pts): The total growth $G = G(p, u)$ of a plant depends on the amount of energy p that the plant produces via photosynthesis, and the amount of energy u that the plant uses to grow. In turn, each of these energy amounts depend on the quantities of water w, oxygen o, and minerals m that are available for the plant, that is,

$$p = p(w, o, m) \qquad \text{and} \qquad u = u(w, o, m).$$

derivative

Find formulas for the rates of change of total plant growth with respect to variations in each of water, oxygen, and mineral availability, that is, compute formulas for:
(Use a diagram showing how these quantities depend on each other!)



Chain Rule

$$\begin{cases} \dfrac{\partial G}{\partial w} = \dfrac{\partial G}{\partial p} \cdot \dfrac{\partial p}{\partial w} + \dfrac{\partial G}{\partial u} \dfrac{\partial u}{\partial w} \\[2ex] \dfrac{\partial G}{\partial o} = \dfrac{\partial G}{\partial p} \dfrac{\partial p}{\partial o} + \dfrac{\partial G}{\partial u} \dfrac{\partial u}{\partial o} \\[2ex] \dfrac{\partial G}{\partial m} = \dfrac{\partial G}{\partial p} \dfrac{\partial p}{\partial m} + \dfrac{\partial G}{\partial u} \dfrac{\partial u}{\partial m} \end{cases}$$

#2 Consider the function $f(x,y) = x^3 - x^2 + x^2y^2 + xy$.

a) Find all critical points of $f(x,y)$

b) Classify them into local min, local max, saddle.

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

a) $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(3x^2 - 2x + 2xy^2 + y, 2x^2y + x\right)$

$\nabla f(x,y) = (0,0) \iff \begin{cases} 3x^2 - 2x + 2xy^2 + y = 0 \\ 2x^2y + x = 0 \Rightarrow x(2xy+1) = 0 \end{cases}$

If $x = 0$: $y = 0$

Get 1 critical point: $(0,0)$

If $x \neq 0$, then $2xy + 1 = 0$, i.e. $2xy = -1$

$y = -\frac{1}{2x}$

From first eqn:

$$3x^2 - 2x + 2x\left(-\frac{1}{2x}\right)^2 - \frac{1}{2x} = 0$$

$$3x^2 - 2x + \frac{\cancel{2x}}{_2\cancel{4x^2}} - \frac{1}{2x} = 0$$

$$3x^2 - 2x + \cancel{\frac{1}{2x}} - \cancel{\frac{1}{2x}} = 0$$

$$x(3x-2) = 0 \xRightarrow{x \neq 0} x = \frac{2}{3} \text{ so } y = -\frac{1}{2 \cdot \frac{2}{3}} = -\frac{3}{4}$$

Get crit pt $\left(\frac{2}{3}, -\frac{3}{4}\right)$.

$\boxed{\text{a) Crit. points: } (0,0), \left(\frac{2}{3}, -\frac{3}{4}\right).}$

b) $$Df(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(3x^2 - 2x + 2xy^2 + y,\ 2x^2y + x\right)$$

$$(\text{Hess } f)(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x - 2 + 2y^2 & 4xy + 1 \\ 4xy + 1 & 2x^2 \end{pmatrix}$$

$$(\text{Hess } f)(0,0) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \qquad \det (\text{Hess } f)(0,0) = -1 < 0$$

$\Longrightarrow$ $\boxed{(0,0) \text{ is } \underline{\text{a saddle}}}$

$$(\text{Hess } f)\left(\frac{2}{3}, -\frac{3}{4}\right) = \begin{pmatrix} 4 - 2 + 2\frac{9}{16} & -2+1 \\ -2+1 & 2 \cdot \frac{4}{9} \end{pmatrix}$$

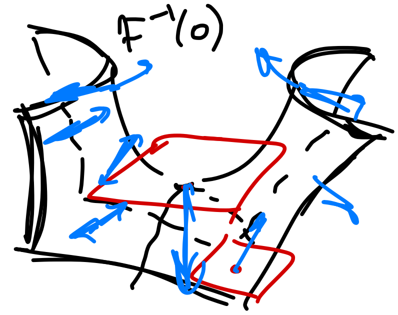
$$= \begin{pmatrix} 2 + \frac{9}{8} & -1 \\ -1 & \frac{8}{9} \end{pmatrix} = \begin{pmatrix} \frac{25}{8} & -1 \\ -1 & 8/9 \end{pmatrix}$$

$$\det (\text{Hess } f)\left(\frac{2}{3}, -\frac{3}{4}\right) = \frac{25}{8} \cdot \frac{8}{9} - 1 = \frac{25}{9} - 1 > 0$$

$$\frac{\partial^2 f}{\partial x^2}\left(\frac{2}{3}, -\frac{3}{4}\right) = \frac{25}{8} > 0$$

$\Longrightarrow$ $\boxed{\left(\frac{2}{3}, -\frac{3}{4}\right) \text{ is a } \underline{\text{local minimum}}.}$

#3 Find the tangent plane to $z = 2x^2 - 3y^2$ at the points $(0,0,0)$ and $(1,1,-1)$.



To find tangent planes use

levelsets: $F(x,y,z) = 2x^2 - 3y^2 - z$

The given surface $z = 2x^2 - 3y^2$ is the 0-levelset of F.

The normal vector to this 0-levelset is ∇F

$$\nabla F(x,y,z) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(4x, -6y, -1\right).$$

At the point $(0,0,0)$: $\nabla F(0,0,0) = (0,0,-1)$.

Equation of tangent plane at $(0,0,0)$: $\langle (x-0, y-0, z-0), \nabla F(0,0,0)\rangle = 0$

$$\langle (x,y,z), (0,0,-1)\rangle = 0$$

$$\boxed{z = 0}$$

Equation of tangent plane at $(1,1,-1)$: $\nabla F(1,1,-1) = (4,-6,-1)$.

$$\langle (x-1, y-1, z+1), (4,-6,-1)\rangle = 0$$

$$4(x-1) - 6(y-1) - (z+1) = 0$$

$$4x - 4 - 6y + 6 - z - 1 = 0$$

$$\boxed{4x - 6y - z + 1 = 0}$$