MAT 226 Lecture 18 4/1/2020

(Lecture 17 was the Midterm Exam, on 3/31/2020)

Lagrange Multipliers:

Q: What for?

A: Constrained optimization! ← Find min/max of functions with certain constraints

Example: Find the maximum and minimum values that $f(x,y) = x^2 + 4y^2 + 2y$ assumes among the points that satisfy $x^2 + y^2 = 1$.

Constraint: $x^2 + y^2 = 1 \rightsquigarrow g(x,y) = x^2 + y^2 - 1$

Target function: $f(x,y) = x^2 + 4y^2 + 2y$

1. Compute gradients of $f(x,y)$ and $g(x,y)$:

$$\nabla f(x,y) = (2x, 8y+2), \quad \nabla g(x,y) = (2x, 2y)$$

2. Solve the equation $\nabla f = \lambda \nabla g$ where $\lambda \in \mathbb{R}$

Lagrange multiplier

$$\begin{cases} 2x = \lambda \cdot 2x \implies 2(\lambda - 1)x = 0 \implies \lambda = 1 \text{ or } x = 0 \\ 8y + 2 = \lambda \cdot 2y \end{cases}$$

If $\lambda = 1$: $8y + 2 = 1 \cdot 2y \Rightarrow 6y = -2 \Rightarrow \boxed{y = -\frac{1}{3}}$

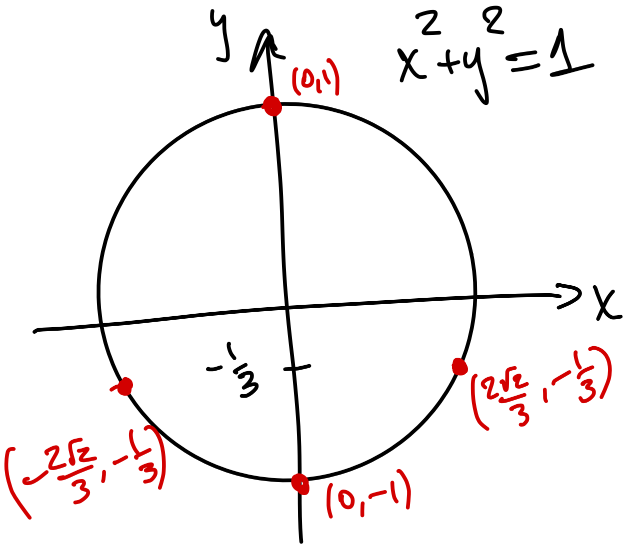
Hence, from the constraint: $x^2 + y^2 = 1$

$$\Rightarrow x^2 + \frac{1}{9} = 1 \Rightarrow x^2 = \frac{8}{9}$$

$$\Rightarrow \boxed{x = \pm \frac{2\sqrt{2}}{3}}$$

Candidate points: $\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$, $\left(-\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$. (so far)

If $x = 0$; from the constraint: $x^2 + y^2 = 1 \Rightarrow y^2 = 1$
$\Rightarrow y = \pm 1$.



Candidate points: $(0,1)$, $(0,-1)$.

Altogether, there are 4 candidate points:

$$\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right), \left(-\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right), (0,1), (0,-1).$$

3. Compute target function ($f(x,y)$) at each of the candidate points. The largest value is the (constrained) maximum and the smallest value is the (constrained) minimum.

$f\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right) = \frac{8}{9} + 4\left(\frac{1}{9}\right) - \frac{2}{3} = \boxed{\frac{2}{3}}$

$f\left(-\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right) = \boxed{\frac{2}{3}}$ ← minimum

$f(0,1) = 4(1)^2 + 2 = \boxed{6}$ ← maximum

$f(0,-1) = 4(-1)^2 - 2 = 2$

Answer: The maximum value is 6 (assumed at $(0,1)$)
The minimum value is $\frac{2}{3}$ (assumed at $\left(\pm\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$)

Summary of Lagrange Multiplier Method:

0. $f(x,y)$ = target function
   $g(x,y) = c$ constraint

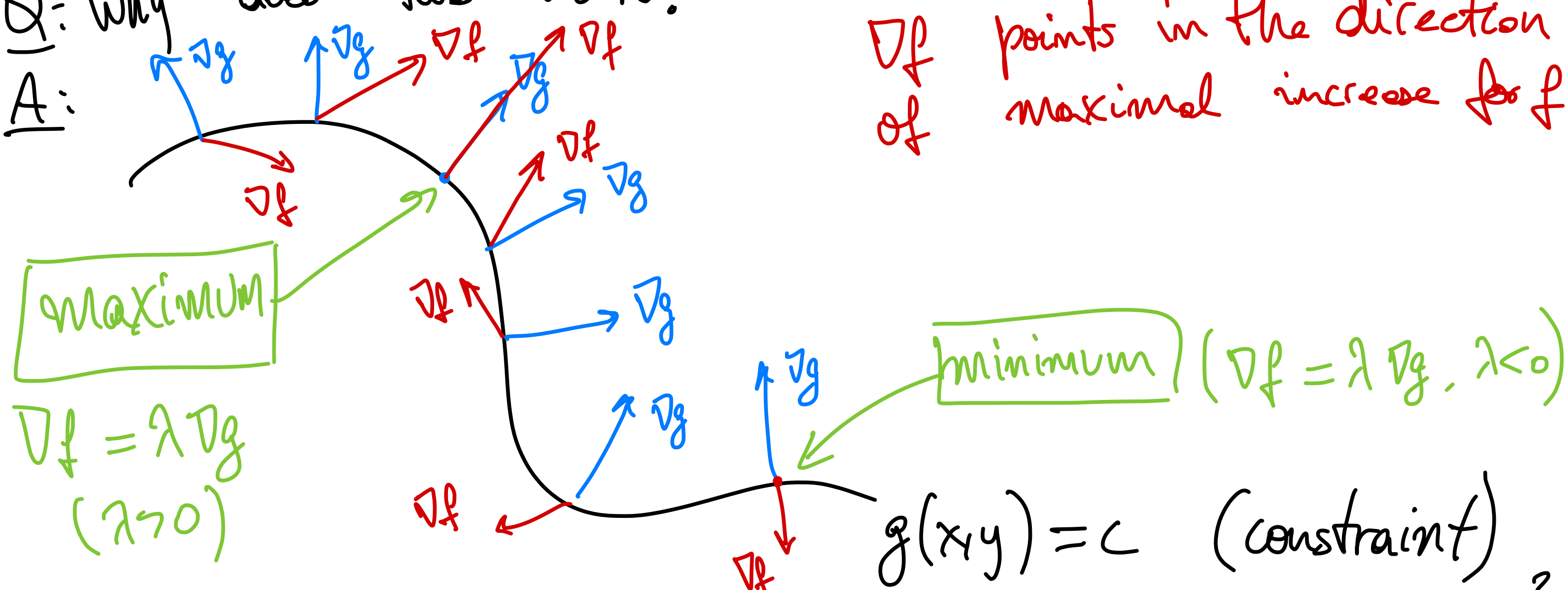
1. Compute $\nabla f(x,y)$, $\nabla g(x,y)$

2. Solve the equation $\nabla f(x,y) = \lambda \nabla g(x,y)$, $\lambda \in \mathbb{R}$
(find all (x,y) that satisfy it for some $\lambda \in \mathbb{R}$)
"Candidate Points"

3. Compute f at each of the candidate points to find largest/smallest values (these are the constrained min/max).

Q: Why does this work?

A:



∇f points in the direction of maximal increase for f

Example from Business: Cobb-Douglas production function

for a company is $f(x,y) = 100\, x^{3/4}\, y^{1/4}$, where "output elasticities"

$$\begin{cases} x = \text{units of labor } (\$150/\text{unit}) \\ y = \text{units of capital } (\$250/\text{unit}) \end{cases}$$

The total expenditures are limited to $\$50{,}000$/year. How many units of labor and capital maximize production?

Target function: $f(x,y) = 100\, x^{3/4}\, y^{1/4}$

Constraint: $\underbrace{150x + 250y}_{g(x,y)} = 50{,}000$

1. $$\nabla f(x,y) = \left(100\, \frac{3}{4}\, x^{-1/4}\, y^{1/4},\ 100\, x^{3/4}\, \frac{1}{4}\, y^{-3/4}\right)$$

$$= \left(75\, x^{-1/4}\, y^{1/4},\ 25\, x^{3/4}\, y^{-3/4}\right).$$

$$\nabla g(x,y) = (150,\ 250)$$

2. $\nabla f = \lambda \nabla g$

$$\begin{cases} 75\, x^{-1/4}\, y^{1/4} = \lambda \cdot 150 \Rightarrow \lambda = \dfrac{75}{150}\, x^{-1/4}\, y^{1/4} = \dfrac{x^{-1/4}\, y^{1/4}}{2} \\ 25\, x^{3/4}\, y^{-3/4} = \lambda \cdot 250 \end{cases}$$

Substitute $\lambda = \dfrac{x^{-1/4}y^{1/4}}{2}$ into the 2^{nd} equ:

$$\cancel{25}\, x^{3/4} y^{-3/4} = \frac{x^{-1/4}y^{1/4}}{2} \cdot \cancel{250} = 5x^{-1/4}y^{1/4}$$

$$x^{3/4+1/4} y^{-3/4} = 5x^{1/4-1/4} y^{1/4} \Rightarrow x y^{-3/4} = 5 y^{1/4}$$

$$\Rightarrow \boxed{x = 5y}$$ ← Relation between x and y needed

To find values, use constraint:

Constraint: $\underbrace{150x + 250y}_{g(x,y)} = 50{,}000$

$$150(5y) + 250y = 50{,}000 \Rightarrow \dots \Rightarrow \boxed{\begin{array}{l} y = 50 \\ x = 250 \end{array}}$$

solve this linear equation!

Candidate point: $(250, 50)$

3. Compute: $f(250,50) = 100 \cdot (250)^{3/4} \cdot 50^{1/4}$ is the maximum.

Answer: $\left\{\begin{array}{l} 250 \text{ units of labor} \\ 50 \text{ units of capital} \end{array}\right.$