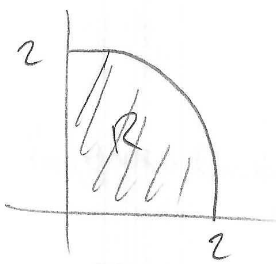
Today: Change of Variables.



$$x^2+y^2 \le 4, \quad x,y \ge 0.$$

$$\iint_R x+y \, dA = \int_0^2 \int_0^{\sqrt{4-x^2}} x+y \, dy \, dx$$

$$= \int_0^2 \left(xy + \frac{y^2}{2} \right) \Big|_0^{\sqrt{4-x^2}} dx = \int_0^2 x\sqrt{4-x^2} + \frac{4-x^2}{2} \, dx$$

$$= -\frac{(4-x^2)^{3/2}}{3} + 2x - \frac{x^3}{6} \Big|_0^2 = 4 - \frac{8}{6} - \left(-\frac{8}{3}\right)$$

$$= 4 - \frac{4}{3} + \frac{8}{3} = 4 + \frac{4}{3} = \frac{12+4}{3} = \boxed{\frac{16}{3}}$$

In polar coordinates. (r,θ)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$dx = \cos\theta \, dr + r(-\sin\theta) \, d\theta$$
$$dy = \sin\theta \, dr + r\cos\theta \, d\theta$$

$$dA = dx\,dy = r\cos^2\theta \, dr\,d\theta - r\sin^2\theta \, d\theta\,dr$$
$$= r(\cos^2\theta + \sin^2\theta) \, dr\,d\theta = \underline{r\,dr\,d\theta}.$$

$$\boxed{\iint_R f(x,y) \, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) \, r\,dr\,d\theta}$$

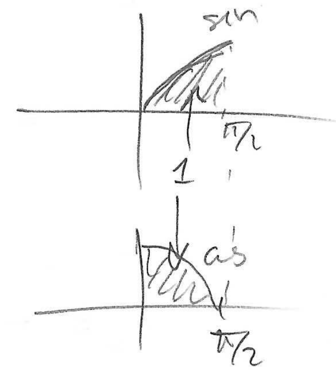
$$\Rightarrow \iint_R x+y\,dA = \int_0^{\pi/2}\int_0^{2} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

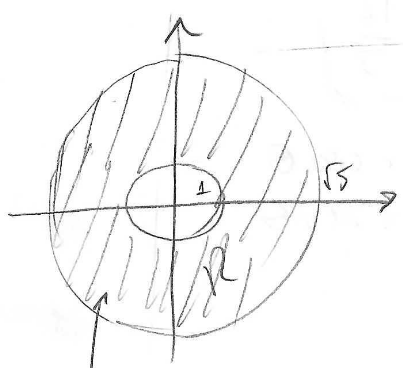
$$= \int_0^{\pi/2}\int_0^{2} r^2(\cos\theta + \sin\theta)\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3}(\cos\theta + \sin\theta)\Big|_0^2\,d\theta = \int_0^{\pi/2} \frac{8}{3}(\cos\theta+\sin\theta)\,d\theta$$

$$= \frac{8}{3}\Bigg(\underbrace{\int_0^{\pi/2}\cos\theta\,d\theta}_{1} + \underbrace{\int_0^{\pi/2}\sin\theta\,d\theta}_{1}\Bigg)$$

$$= \boxed{\frac{16}{3}}$$





$$\iint_R (x^2+y)\,dA = \int_0^{2\pi}\int_1^{\sqrt5}(r^2\cos^2\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt5}\left(r^3\cos^2\theta + r^2\sin\theta\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^4}{4}\cos^2\theta + \frac{r^3}{3}\sin\theta\right)\Bigg|_1^{\sqrt5}d\theta$$

$$= \int_0^{2\pi} \frac{25}{4}\cos^2\theta + \frac{5\sqrt5}{3}\sin\theta - \frac14\cos^2\theta - \frac13\sin\theta\;\;d\theta$$

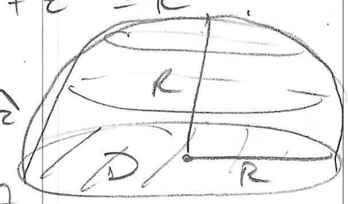
periodic [0,2π]

$$= \int_0^{2\pi} 6\cos^2\theta + \frac{5\sqrt5-1}{3}\sin\theta\,d\theta = 6\int_0^{2\pi}\frac{1+\cos2\theta}{2}\,d\theta = 3.2\pi$$

periodic on [0,2π]

$$= \boxed{6\pi}$$

Find volume inside the sphere $x^2+y^2+z^2=R^2$:

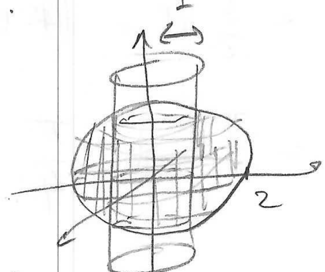


$$2V = \iint_D \sqrt{R^2-x^2-y^2}\, dA = \int_0^{2\pi}\int_0^R \sqrt{R^2-(r\cos\theta)^2-(r\sin\theta)^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^R \sqrt{R^2-r^2}\cdot r\,dr\,d\theta = 2\pi\int_0^R r\sqrt{R^2-r^2}\,dr$$

$$= -2\pi\cdot\frac{(R^2-r^2)^{3/2}}{3}\Bigg|_0^R = +\frac{2\pi}{3}R^3 \Rightarrow \boxed{V = \frac{4}{3}\pi R^3}$$

Find volume inside the sphere $x^2+y^2+z^2=4$ and <u>outside</u> the cylinder $x^2+y^2=\underline{1}$.



$$2V = \int_0^{2\pi}\int_1^2 \sqrt{4-(r\cos\theta)^2-(r\sin\theta)^2}\, r\,dr\,d\theta$$

$$= 2\pi\int_1^2 r\sqrt{4-r^2}\,dr = 2\pi\left(-\frac{(4-r^2)^{3/2}}{3}\right)\Bigg|_1^2 = \frac{2\pi}{3}3^{3/2} = \boxed{2\pi\sqrt{3}}$$

$$\Rightarrow \boxed{V = 4\pi\sqrt{3}}$$

What about the volume inside both cylinder and sphere?

$$2V = \int_0^{2\pi}\int_0^1 r\sqrt{4-r^2}\,dr\,d\theta = 2\pi\left(-\frac{(4-r^2)^{3/2}}{3}\right)\Bigg|_0^1 =$$

$$= -\frac{2\pi}{3}\cdot 3^{3/2} + \frac{2\pi}{3}\cdot 4^{3/2} = \frac{2\pi}{3}\left(8-3^{3/2}\right) = \frac{16\pi}{3} - 2\pi\sqrt{3}$$

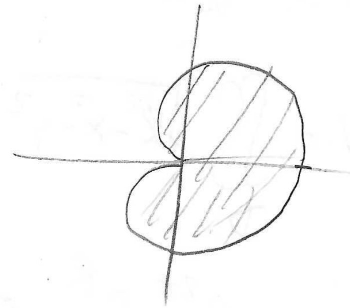
$$\Rightarrow V = \frac{32\pi}{3} - 4\pi\sqrt{3}$$ ✓

$\boxed{\frac{4}{3}\pi 2^3} =$

previous computation

Extra example:

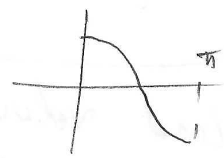
Area of cardioid $r(\theta) = 1 + \cos\theta$



$$\frac{1}{2} A = \int_0^{\pi}\int_0^{1+\cos\theta} 1 \, dA = \int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{r^2}{2}\Big|_0^{1+\cos\theta} d\theta = \int_0^{\pi} \frac{(1+\cos\theta)^2}{2}\, d\theta = \frac{1}{2}\int_0^{\pi} 1 + 2\cos\theta + \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\pi + \int_0^{\pi} \cos\theta \, d\theta + \frac{1}{2}\int_0^{\pi} \cos^2\theta \, d\theta$$



$$= \frac{1}{2}\pi + (1-1) + \frac{1}{2}\int_0^{\pi} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{2\pi + \pi}{4} = \frac{3\pi}{4}$$

$$\Rightarrow A = \frac{6\pi}{4} = \frac{3\pi}{2}$$

$a = \frac{1}{2} A = \frac{6\pi}{4} = \frac{3\pi}{2}$ ✓