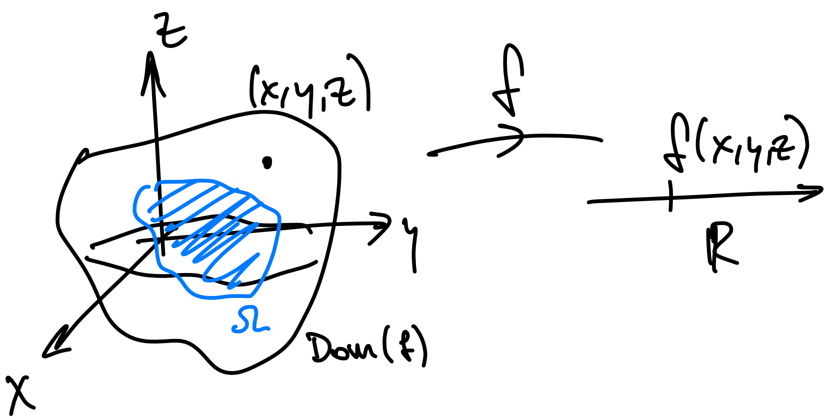
Triple Integrals:

$$f : \mathrm{Dom}(f) \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$f(x,y,z) = 5xyz$$

$$f(x,y,z) = e^x(y+2z)$$

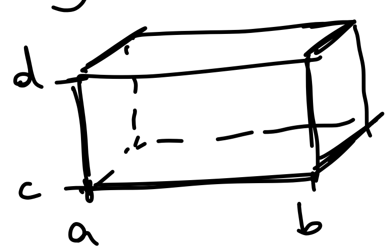
$$f(x,y,z) = \sqrt{4-x^2-y^2-z^2}$$



Suppose $\Omega \subset \mathrm{Dom}(f)$ is parametrized by $\begin{cases} a \le x \le b \\ h_1(x) \le y \le h_2(x) \\ g_1(x,y) \le z \le g_2(x,y) \end{cases}$

$$\iiint_\Omega f \, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z) \, dz \, dy \, dx$$

Simple example: $\Omega = [a,b] \times [c,d] \times [e,f]$



$$\Omega = [1,2] \times [0,1] \times [2,3].$$

$$\iiint_\Omega 5xyz \, dV = \int_1^2 \left(\int_0^1 \left(\int_2^3 5xyz \, dz \right) dy \right) dx$$

$$= \int_1^2 \int_0^1 5xy \frac{z^2}{2} \Big|_2^3 \, dy \, dx$$

$$= \int_1^2 \left(\int_0^1 5xy \left(\underbrace{\frac{9}{2} - 2}_{5/2} \right) dy \right) dx$$

$$= \frac{25}{2} \int_1^2 \left(\int_0^1 xy \, dy \right) dx = \frac{25}{2} \int_1^2 \frac{xy^2}{2} \Big|_0^1 dx$$

$$= \frac{25}{2} \int_1^2 \frac{x}{2} \, dx = \frac{25}{4} \frac{x^2}{2} \Big|_1^2 = \frac{25}{4} \left(2 - \frac{1}{2} \right)$$

$$= \frac{25}{4} \left(\frac{3}{2} \right) = \boxed{\frac{75}{8}}.$$

<u>Note</u>: Since $f(x,y,z) = \phi_1(x)\phi_2(y)\phi_3(z)$ and Ω is a parallelepiped, the integral above can be done "all at once":

$$\int_a^b \int_c^d \left(\int_e^f \underbrace{\phi_1(x)\phi_2(y)}_{\text{constant}} \phi_3(z) \, dz \right) dy\, dx = \int_a^b \int_c \underbrace{\phi_1(x)\phi_2(y)}_{\text{const}} \left[\int_e^f \phi_3(z) \, dz \right] dy\, dx$$

$$= \int_a^b \phi_1(x) \left[\int_c^d \phi_2(y) \underbrace{\left[\int_e^f \phi_3(z) \, dz \right]}_{\text{const}} dy \right] dx =$$

$$= \left[\int_e^f \phi_3(z)\,dz\right] \int_a^b \phi_1(x) \underbrace{\left[\int_c^d \phi_2(y)\,dy\right]}_{\text{const}} dx$$

$$= \left(\int_e^f \phi_3(z)\,dz\right)\left(\int_c^d \phi_2(y)\,dy\right)\left(\int_a^b \phi_1(x)\,dx\right)$$

In the above example

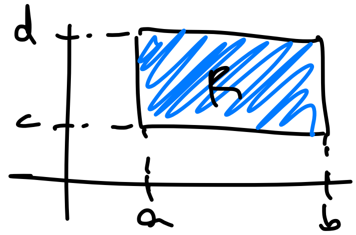
$$\int_1^2\int_0^1\int_2^3 5xyz\,dz\,dy\,dx = 5\left(\int_2^3 z\,dz\right)\left(\int_0^1 y\,dy\right)\left(\int_1^2 x\,dx\right)$$

$$= 5\ \frac{z^2}{2}\Big|_2^3 \cdot \frac{y^2}{2}\Big|_0^1 \cdot \frac{x^2}{2}\Big|_1^2$$

$$= 5\left(\frac{9-4}{2}\right)\left(\frac{1}{2}\right)\left(\frac{4-1}{2}\right) = \frac{5.5.3}{8} = \boxed{\frac{75}{8}}$$

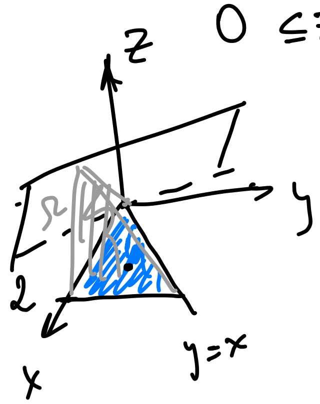
Remark. Same goes for (double) integrals of functions of 2 variables when R is a rectangle and $f(x,y)$ is a product of functions of each individual variable

$$\iint_R f\,dA = \int_a^b\int_c^d \phi_1(x)\,\phi_2(y)\,dy\,dx = \left(\int_a^b \phi_1(x)\,dx\right)\left(\int_c^d \phi_2(y)\,dy\right)$$



A (slightly) more difficult triple integral

$$\Omega: \quad 0 \le x \le 2 \qquad f(x,y,z) = e^x(y+2z)$$
$$0 \le y \le x$$
$$0 \le z \le x+y$$



$$\iiint_\Omega f\,dV = \int_0^2 \int_0^x \int_0^{x+y} e^x(y+2z)\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^x \left(e^x yz + e^x z^2\right)\Big|_0^{x+y} dy\,dx$$

$$= \int_0^2 \int_0^x e^x y(x+y) + e^x(x+y)^2\,dy\,dx$$

$$= \int_0^2 \int_0^x e^x xy + e^x y^2 + e^x x^2 + 2e^x xy + e^x y^2\,dy\,dx$$

$$= \int_0^2 \int_0^x 3e^x xy + 2e^x y^2 + e^x x^2\,dy\,dx$$

$$= \int_0^2 \left(3e^x x\frac{y^2}{2} + 2e^x\frac{y^3}{3} + e^x x^2 y\right)\Big|_0^x dx$$

$$= \int_0^2 \frac{3}{2}e^x x^3 + \frac{2}{3}e^x x^3 + e^x x^3\,dx$$

$$= \int_0^2 \left(\frac{3}{2} + \frac{2}{3} + 1\right) e^x x^3\,dx = \frac{19}{6}\int_0^2 e^x x^3\,dx$$

integration by parts $3x$

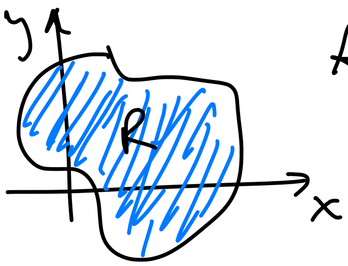
$$= \cdots = \frac{19}{6}\cdot e^x\left(x^3 - 3x^2 + 6x - 6\right)\Big|_0^2$$

$$= \frac{19}{6} e^2 (8 - 12 + 12 - 6) - \frac{19}{6}(-6)$$

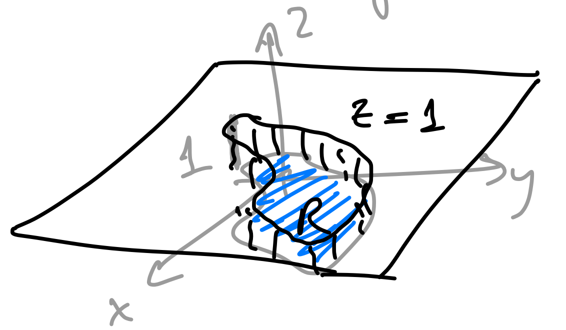
$$= \frac{19}{3} e^2 + 19 = \boxed{19\left(\frac{e^2}{3} + 1\right)}$$

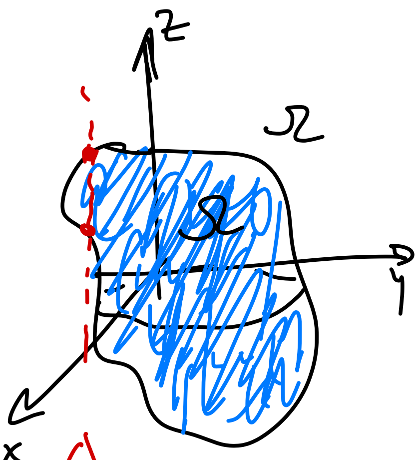
Geometric Application: Areas and Volumes

Remember: $\iint\limits_R f \, dA$ = Volume under the graph $z = f(x,y)$ over the region R.



$$\text{Area}(R) = \iint\limits_R 1 \, dA$$ = Volume under the graph $z = 1$ over the region R





$$\text{Volume}(\Omega) = \iiint\limits_\Omega 1 \, dV$$

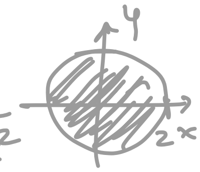
⚠ Sometimes our region Ω is not defined in terms of being "under the graph" of any $f(x,y)$.

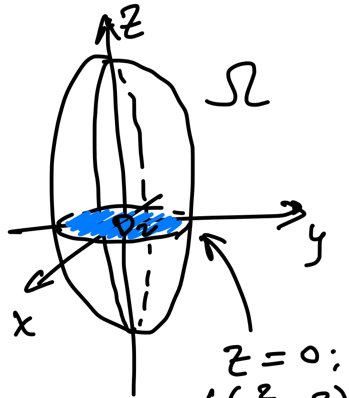
Example: Find the volume inside the ellipsoid

$$4x^2+4y^2+z^2=16.$$

$-2\le x\le 2$

$-\sqrt{4-x^2}\le y\le\sqrt{4-x^2}$





$z=0$:

$4(x^2+y^2)=16$

$x^2+y^2=4$

Ω: $(x,y)\in$ Disk of radius 2 centered at 0. (call it D_2)

$$-\sqrt{16-4(x^2+y^2)}\le z\le\sqrt{16-4(x^2+y^2)}$$

$$\text{Volume}(\Omega)=\iiint_{\Omega}1=2\iint_{D_2}\left[\int_0^{\sqrt{16-4(x^2+y^2)}}1\,dz\right]dA$$

$$=2\iint_{D_2}z\Big|_0^{\sqrt{16-4(x^2+y^2)}}dA=2\iint_{D_2}\sqrt{16-4(x^2+y^2)}\,dA$$

$$=2\int_0^{2\pi}\int_0^2\sqrt{16-4r^2}\;r\,dr\,d\theta=4\pi\int_0^2 r\sqrt{16-4r^2}\,dr$$

in polar coordinates:

$0\le r\le 2$

$0\le\theta\le 2\pi$

$$=4\pi\left.\frac{(16-4r^2)^{3/2}}{3/2\,(-8)}\right|_0^2=-\frac{\pi}{3}(0-16)^{3/2}$$

$$=\boxed{\frac{64\pi}{3}}$$

Note: The above way of computing this triple integral uses "cylindrical coordinates"