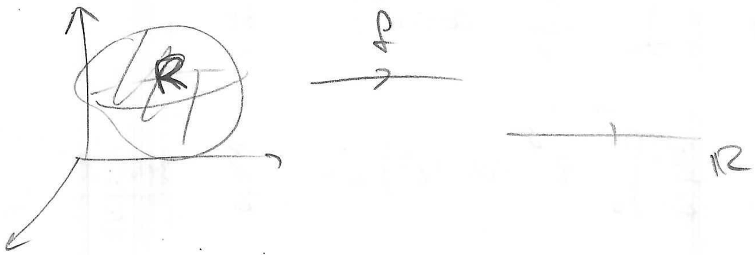
MAT226 Lecture 18 11/11/2019

Triple Integrals. (Sec 14.6)



$$\iiint_R f(x,y,z)\, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

eg. $\longrightarrow$ $\boxed{\text{Easier example first!}}$: $\int_2^3 \int_1^2 \int_0^1 8xyz\, dz\, dy\, dx = 15$

$$\int_0^2 \int_0^x \int_0^{x+y} e^x(y+2z)\, dz\, dy\, dx = \int_0^2 \int_0^x e^x y z + e^x z^2 \Big|_0^{x+y} dy\, dx$$

$$= \int_0^2 \int_0^x e^x y(x+y) + e^x (x+y)^2\, dy\, dx = \int_0^2 \int_0^x e^x (x^2 + 3xy + 2y^2)\, dy\, dx$$

$$= \int_0^2 e^x \left(x^2 y + \frac{3xy^2}{2} + \frac{2y^3}{3} \right) \Big|_0^x dx = \int_0^2 e^x \left(x^3 + \frac{3x^3}{2} + \frac{2x^3}{3} \right) dx$$

$$= \int_0^2 e^x x^3 \left(\frac{6+9+4}{6} \right) dx = \frac{19}{6} \int_0^2 e^x x^3 dx$$

$$= \frac{19}{6}\, e^x (x^3 - 3x^2 + 6x - 6) \Big|_0^2 = 19\left(\frac{e^2}{3} + 1\right).$$

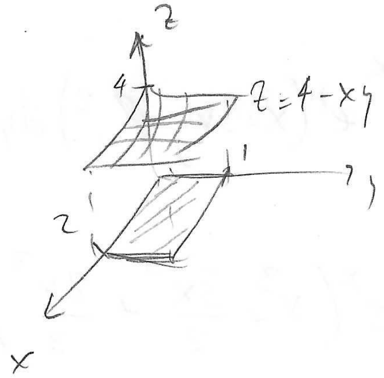
$$\int_0^{\pi}\int_0^{z^2}\int_0^{3} y\cos(z^5)\,dx\,dy\,dz = \int_0^{\pi}\int_0^{z^2} xy\cos(z^5)\Big|_0^3\,dy\,dz$$

$$= \int_0^{\pi}\int_0^{z^2} 3y\cos(z^5)\,dy\,dz = \int_0^{\pi} \frac{3y^2}{2}\cos(z^5)\Big|_0^{z^2}\,dz =$$

$$= \int_0^{\pi} \frac{3z^4}{2}\cos(z^5)\,dz = \frac{3}{2}\int_0^{\pi} z^4\cos(z^5)\,dz = \frac{3}{2}\,\frac{\sin(z^5)}{5}\Big|_0^{\pi}$$

$$= \boxed{\frac{3}{10}\sin(\pi^5)}$$

$\iiint_R 3-4x\,dV$ where R is region below $z=4-xy$ and above the region in xy-plane defined by $0 \le x \le 2,\ 0 \le y \le 1$



$$\iiint_R 3-4x\,dV = \int_0^2\int_0^1\int_0^{4-xy} 3-4x\,dz\,dy\,dx$$

$$= \int_0^2\int_0^1 (3-4x)z\Big|_0^{4-xy}\,dy\,dx = \int_0^2\int_0^1 (4-xy)(3-4x)\,dy\,dx$$

$$= \int_0^2\int_0^1 12-16x-3xy+4x^2y\,dy\,dx = \int_0^2 12y-16xy-\frac{3xy^2}{2}+\frac{4x^2y^2}{2}\Big|_0^1\,dx$$

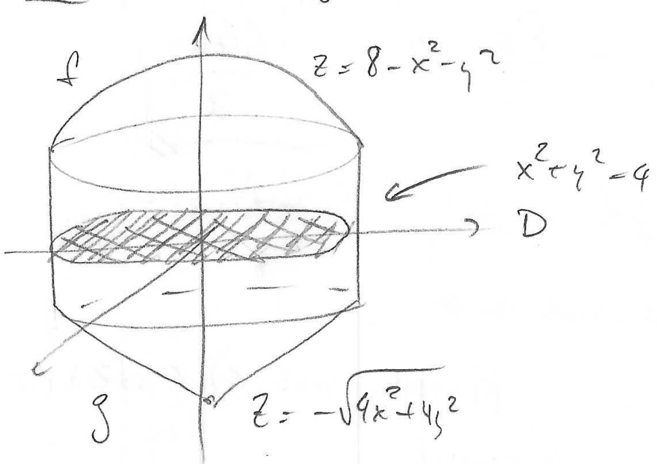
$$= \int_0^2 12 - 16x - \frac{3x}{2} + \frac{4x^2}{2}\,dx = \int_0^2 12 - \frac{32+3}{2}x + 2x^2\,dx$$

$$= 12x - \frac{35}{2}\frac{x^2}{2} + \frac{2x^3}{3}\Big|_0^2 = 24 - \frac{35}{4}\cdot 4 + \frac{2\cdot 8}{3} = 24 - 35 + \frac{16}{3}$$

$$= -11 + \frac{16}{3} = \frac{-33+16}{3} = \boxed{-\frac{17}{3}}$$

$$\boxed{\underline{\text{Note}}:\quad Vol(R) = \iiint_R 1\,dV}$$

$\underline{\text{EX}}$: Find volume under $z = 8 - x^2 - y^2$ above $z = -\sqrt{4x^2+4y^2}$ inside the cylinder $x^2+y^2=4$.



$$\iiint_R dV = \iint_D \left(\int_{-\sqrt{4x^2+4y^2}}^{8-x^2-y^2} 1\,dz \right) dA$$

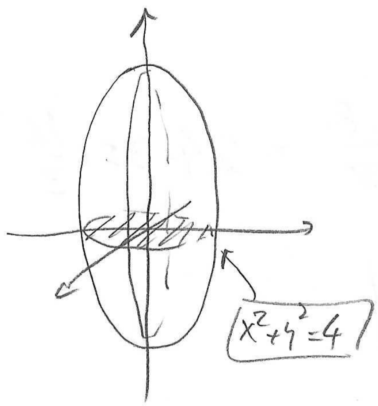
$$= \iint_D \underbrace{8 - x^2 - y^2}_{f} + \underbrace{\sqrt{4x^2+4y^2}}_{-g}\,dA = \int_0^{2\pi}\int_0^2 8 - r^2 + \sqrt{4r^2}\,r\,dr\,d\theta$$

polar coord.

$$= 2\pi \int_0^2 (8 - r^2 + 2r)\,r\,dr = 2\pi \int_0^2 8r - r^3 + 2r^2\,dr = 2\pi \left(4r^2 - \frac{r^4}{4} + \frac{2r^3}{3}\right)\Big|_0^2$$

$$= 2\pi\left(16 - 4 + \frac{16}{3}\right) = 2\pi\left(12 + \frac{16}{3}\right) = 2\pi\,\frac{36+16}{3} = \boxed{\frac{104\pi}{3}}$$

Volume of Ellipsoid:



$$4x^2+4y^2+z^2=16$$

$$V=\iiint_E dV = 2\iint_D \int_0^{\sqrt{16-4x^2-4y^2}} dz\, dA$$

$$= 2\iint_D \sqrt{16-4x^2-4y^2}\, dA$$

polar

$$= 2\int_0^{2\pi}\int_0^2 \sqrt{16-4r^2}\, r\,dr\,d\theta = 4\pi\int_0^2 2r\sqrt{4-r^2}\,dr$$

$$= -4\pi \left.\frac{(4-r^2)^{3/2}}{3/2}\right|_0^2 = \frac{8\pi}{3}\cdot 4^{3/2} = \boxed{\frac{64\pi}{3}}$$

These are integrals in <u>cylindrical coordinates</u>:

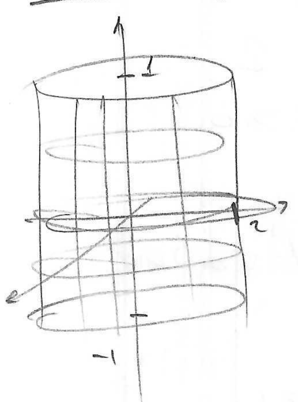
$$\iiint_R f(x,y,z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\,dr\,d\theta$$

$$\left(dV = r\,dz\,dr\,d\theta \right)$$

<u>cylindrical coord.:</u>
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \text{ polar in } xy\text{-plane}, \qquad z = z$$

$$\iiint_R f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta,\,r\sin\theta)}^{h_2(r\cos\theta,\,r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,\underbrace{r\,dz\,dr\,d\theta}_{dV}$$

Ex: Integrate $f(x,y,z) = x^2z^2 + y^2z^2$ on region R which is cylinder $x^2+y^2 \le 4$, $-1 \le z \le 1$.

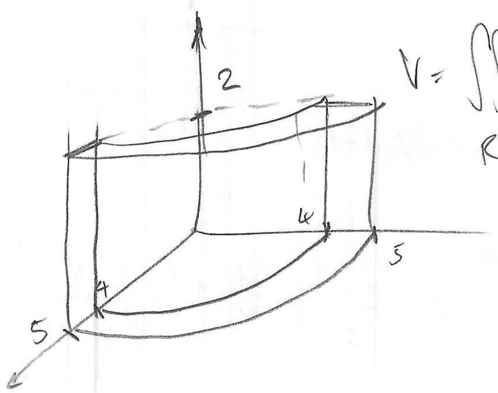


$$\iiint_R f\,dV = \int_0^{2\pi}\int_0^2\int_{-1}^1 r^2z^2\,r\,dz\,dr\,d\theta$$

$$= 2\pi\int_0^2 r^3\,dr\int_{-1}^1 z^2\,dz =$$

$$= 2\pi\cdot\frac{2^4}{4}\cdot\frac{2}{3} = \boxed{\frac{16\pi}{3}}$$

Ex: Find volume of cylindrical shell below:

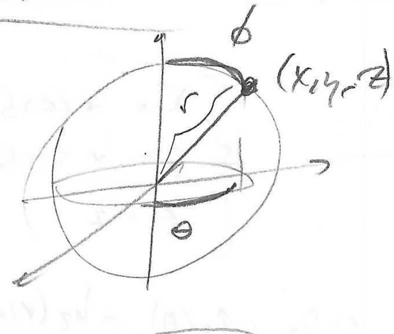


$$V = \iiint_R dV = \int_0^{\pi/2}\int_4^5\int_0^2 r\,dz\,dr\,d\theta$$

$$= \frac{\pi}{2}\cdot 2\,\frac{r^2}{2}\Big|_4^5 = \frac{\pi}{2}(25-16) = \boxed{\frac{9\pi}{2}}$$

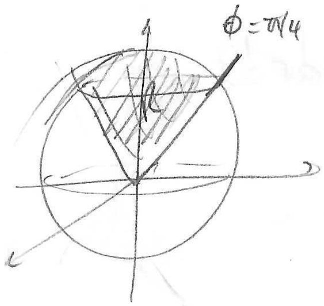
Integrals in spherical coord..

$$\left.\begin{aligned} x &= r\cos\theta \sin\phi \\ y &= r\sin\theta \sin\phi \\ z &= r\cos\phi \end{aligned}\right]$$



$$\iiint_R f(x,y,z)\, dV = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(\theta,\phi)}^{h_2(\theta,\phi)} f(\cdot,\cdot,\cdot)\, r^2 \sin\phi \cdot dr\, d\phi\, d\theta$$

Ex: Find the volume inside sphere of radius 2 which is inside the cone $x^2+y^2=z^2$, $z \geq 0$



$$Vol(R) = \iiint_R dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^2 r^2 \sin\phi\, dr\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi/4} \sin\phi \cdot d\phi \int_0^2 r^2\, dr$$

$$= 2\pi \left(-\cos\phi\right)\Big|_0^{\pi/4} \cdot \frac{r^3}{3}\Big|_0^2 = \frac{16\pi}{3}\left(-\frac{\sqrt{2}}{2}+1\right)$$

$$= \boxed{\frac{8\pi}{3}\left(2-\sqrt{2}\right)}$$

How about vol. outside the cone?

$$Vol = \iiint_\Omega dV = \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^2 r^2 \sin\phi\, dr\, d\phi\, d\theta$$

$$= 2\pi \frac{r^3}{3}\Big|_0^2 \cdot \left(-\cos\phi\right)\Big|_{\pi/4}^{3\pi/4} = \frac{16\pi}{3}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right) = \boxed{\frac{16\pi\sqrt{2}}{3}}$$

Vol of sphere: $Vol = \int_0^{2\pi}\int_0^{\pi}\int_0^R r^2 \sin\phi\, dr\, d\phi\, d\theta = 2\pi \frac{R^3}{3}(1+1) = \frac{4\pi R^3}{3}$