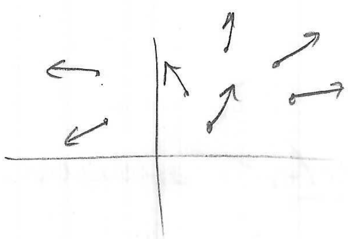
MAT226 Lecture 20, 21 11/18/2019
11/20/2019

Vector fields (Sec 15.1)

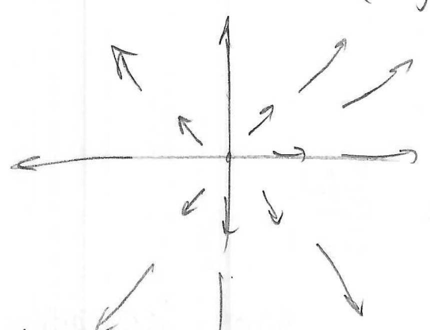
$F: \mathbb{R}^2 \to \mathbb{R}^2$



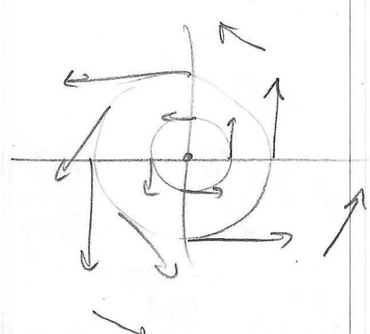
$$F(x,y) = (f_1(x,y), f_2(x,y)) = (M(x,y), N(x,y))$$

$F: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad F(x,y,z) = (f_1(x,y,z), \_\_\_) = (M, N, P)$

Examples: $F(x,y) = (x,y)$



$G(x,y) = (-y, x)$



→ computer DEMOS

Def: (Flow line / Integral curve): γ is int. curve of F if $\boxed{\gamma'(t) = F(\gamma(t)) \ \forall t}$

$f: \mathbb{R}^2 \to \mathbb{R}$, $F = \nabla f$ is a vector field

e.g.: $f(x,y) = \frac{1}{2}(x^2+y^2)$ $\quad F = \nabla f$ is the above.

Q: How about G? Is $G = \nabla g$ for some g?

A: No: Periodic orbits! Function increases along integral curves of ∇f. So can't come back to where it was unless it stayed constant. But then this would be a levelset (and that's orthogonal)

Def: A vector field $F: \mathbb{R}^n \to \mathbb{R}^n$ is <u>conservative</u> if $\exists f: \mathbb{R}^n \to \mathbb{R}$ such that $F = \nabla f$. The function f is called a <u>potential function</u> for F.

Q: When is $F = (M, N)$ conservative?

Thm: Let $\Omega \subset \mathbb{R}^2$ be a <u>simply-connected domain</u>. Then $F: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is <u>conservative</u> if and only if
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Ex:

$F(x,y) = (x^2 y, xy)$ $\quad \frac{\partial M}{\partial y} = x^2 \neq \frac{\partial N}{\partial x} = y$ <u>not conservative.</u>

$F(x,y) = (2xy, x^2 - y)$ $\quad \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$ <u>conservative</u>

$F(x,y) = (2x, y)$ $\quad \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$ <u>conservative!</u>

Q: If F is conservative, how to find f s.t. $F = \nabla f$?

A: $f(x,y) = \int M(x,y)\,dx + g(y)$

or $f(x,y) = \int N(x,y)\,dy + g(x)$.

Ex. Find a potential function for

$$F(x,y) = (2xy,\ x^2-y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$f(x,y) = \int 2xy\, dx + g(y) = x^2y + g(y)$$

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - y = N$$

$$\Downarrow$$

$$g'(y) = -y \Rightarrow g(y) = -\frac{y^2}{2} + c$$

$$\Rightarrow \boxed{f(x,y) = x^2y - \frac{y^2}{2} + c}$$

Ex: $f(x,y) = e^x \cos y + K$.
$F(x,y) = (e^x \cos y,\ -e^x \sin y)$

Q: When is $F : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ conservative? $F = (M, N, P)$

Thm: If $\Omega \subset \mathbb{R}^3$ is simply-connected, then $F : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is conservative if and only if curl $F = 0$.

$$\text{curl } F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Ex.: $F(x,y,z) = (2xy,\ x^2+z^2,\ 2yz)$

$\nabla \times F = 0$

Find a potential?

$f(x,y,z) = \int 2xy\, dx + g(y,z)$

$= x^2 y + g(y,z)$

$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y}(y,z) = x^2 + z^2 \Rightarrow \frac{\partial g}{\partial y} = z^2 \Rightarrow$

$\Rightarrow g(y,z) = z^2 y + h(z)$

So $f(x,y,z) = x^2 y + z^2 y + h(z)$

$\frac{\partial f}{\partial z} = 2zy + h'(z) = 2yz \Rightarrow h'(z) = 0 \Rightarrow h(z) = c.$

$\boxed{f(x,y,z) = (x^2+z^2)\,y + C}$

Ex: $F(x,y,z) = (ye^{xy},\ xe^{xy} + \sin z,\ y\cos z)$

$f(x,y,z) = e^{xy} + y\sin(z) + C$

$\int ye^{xy}\, dx = e^{xy} + g(y,z)$

$xe^{xy} + \frac{\partial g}{\partial y} = xe^{xy} + \sin z$

$\Rightarrow g = y\sin z + h(z)$

✓

Divergence: $\operatorname{div} F = \nabla \cdot F = \left\langle \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), (M,N,P) \right\rangle$

$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$

Measures how <u>volume</u> changes along a flow.

Ex: $\vec{F}(x,y,z) = (x^3 y^2,\ x z^2,\ x^2 y)$

$\operatorname{div} \vec{F} = 3x^2 y^2 z$

incompressible fluid $\Leftrightarrow \operatorname{div} \vec{F} = 0$