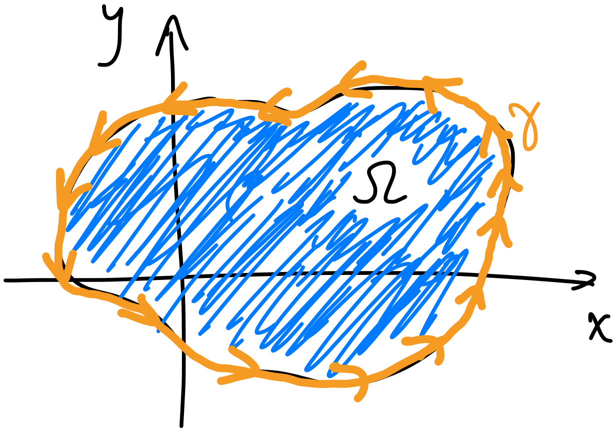
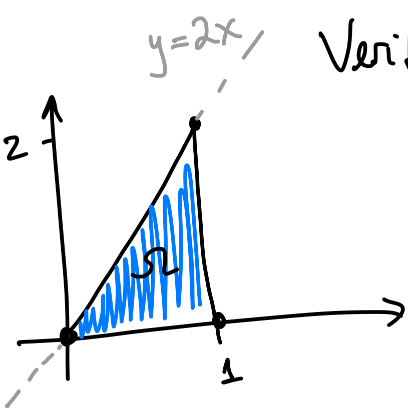
Green's Theorem: Suppose $\Omega \subset \mathbb{R}^2$ is simply-connected and denote by γ its boundary curve, oriented counterclockwise. If $\vec{F}: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{F} = (M, N)$ is a smooth vector field, then



$$\underbrace{\int_\gamma \vec{F}\, d\gamma}_{\text{line integral of } \vec{F} \text{ on } \gamma} = \underbrace{\iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA}_{\text{double integral of } f(x,y) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \text{ over } \Omega.}$$

Example: $\vec{F}(x,y) = (\underbrace{xy}_{=M}, \underbrace{x^2y^3}_{=N})$, Ω = triangle with vertices $(0,0), (1,0), (1,2)$.



Verify that both integrals in Green's Theorem agree

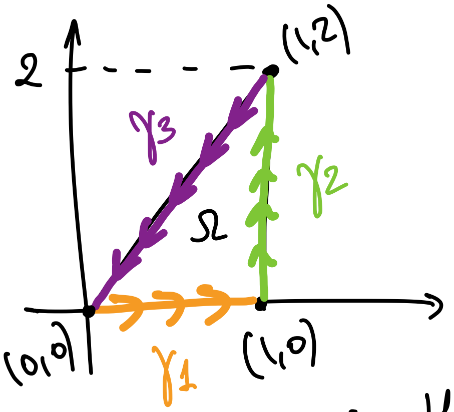
1. RHS. Parametrize Ω: $0 \le x \le 1$, $0 \le y \le 2x$

$$\iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^1 \int_0^{2x} 2xy^3 - x \, dy\, dx$$

$$= \int_0^1 \left(\frac{\cancel{2}xy^4}{\cancel{4}\,2} - xy\right)\Bigg|_0^{2x} dx = \int_0^1 \frac{x}{2}(2x)^4 - x(2x)\, dx = \int_0^1 8x^5 - 2x^2 dx$$

$$= \frac{8x^6}{6} - \frac{2x^3}{3}\Bigg|_0^1 = \frac{4}{3} - \frac{2}{3} = \boxed{\frac{2}{3}}$$

② LHS: Boundary of Ω consists of 3 line segments



γ_1: $\gamma_1(t) = (1-t)(0,0) + t(1,0) = (t,0)$, $t\in[0,1]$

γ_2: $\gamma_2(t) = (1-t)(1,0) + t(1,2) = (1,2t)$, $t\in[0,1]$

γ_3: $\gamma_3(t) = (1-t)(1,2) + t(0,0) = (1-t, 2-2t)$, $t\in[0,1]$

and γ is the concatenation of the above 3 curves

$$\int_\gamma \vec{F}\, d\gamma = \int_{\gamma_1} \vec{F}\, d\gamma_1 + \int_{\gamma_2} \vec{F}\, d\gamma_1 + \int_{\gamma_3} \vec{F}\, d\gamma_3 = 0 + 4 - \frac{10}{3} = \boxed{\frac{2}{3}}$$

$\underline{\gamma_1:}$ $\vec{F}(\gamma_1(t)) = (0,0)$ $\qquad \displaystyle\int_{\gamma_1} \vec{F}\, d\gamma_1 = \int_0^1 \underbrace{\langle (0,0), (1,0) \rangle}_{=0}\, dt = \underline{\underline{0}}$

$\gamma_1'(t) = (1,0)$

$\underline{\gamma_2:}$ $\vec{F}(\gamma_2(t)) = (2t, 8t^3)$ $\qquad \displaystyle\int_{\gamma_2} \vec{F}\, d\gamma_2 = \int_0^1 \langle (2t, 8t^3), (0,2) \rangle\, dt =$

$\gamma_2'(t) = (0,2)$

$$= \int_0^1 16t^3\, dt = 4t^4 \Big|_0^1 = \underline{\underline{4}}.$$

$\underline{\gamma_3:}$ $\vec{F}(\gamma_3(t)) = \left((1-t)(2-2t), (1-t)^2(2-2t)^3\right) = \left(2(1-t)^2, 8(1-t)^5\right).$

$\gamma_3'(t) = (-1,-2).$

$$\int_{\gamma_3} \vec{F}\, d\gamma_3 = \int_0^1 \langle \left(2(1-t)^2, 8(1-t)^5\right), (-1,-2) \rangle\, dt$$

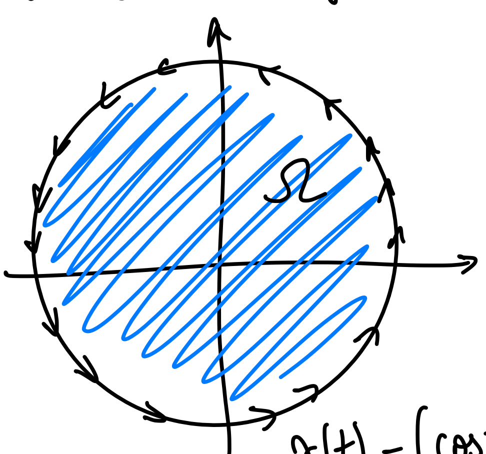
$$= \int_0^1 -2(1-t)^2 - 16(1-t)^5 \, dt \qquad \begin{aligned} u &= 1-t \\ du &= -dt \end{aligned}$$

$$= \int_1^0 +2u^2 + 16u^5 \, du = \frac{2u^3}{3} + \frac{16u^6}{6} \Big|_1^0$$

$$= -\frac{2}{3} - \frac{8}{3} = -\frac{10}{3}.$$

Combining these we see that LHS also equals $2/3$.

<u>Ex</u>: Use Green's Thm to compute the work performed by the force $\vec{F}(x,y) = (\underbrace{xy+1}_{=M}, \underbrace{y^2 + e^y \sin^5(y^{2020} + 17y^4) - 3}_{=N})$ on a particle that completes 1 revolution around the origin along a circle of radius 1; counterclockwise



$\gamma(t) = (\cos t, \sin t)$

$t \in [0, 2\pi]$.

$$W = \int_\gamma \vec{F} \, d\gamma \overset{\text{Green's Thm}}{=} \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA =$$

$$= \iint_\Omega (0 - x) \, dA = -\iint_\Omega x \, dA$$

$$= -\int_0^{2\pi} \int_0^1 r\cos\theta \; r \, dr \, d\theta$$

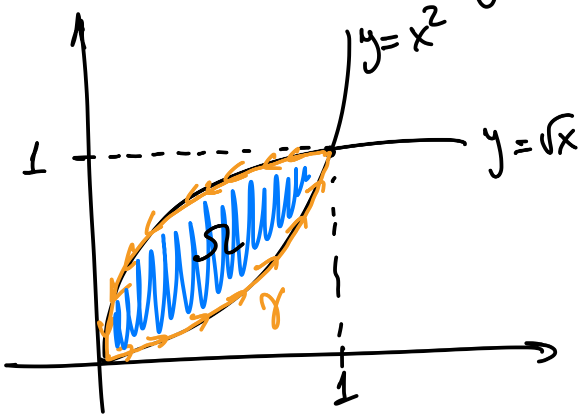
$$= -\underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{=0} \cdot \underbrace{\int_0^1 r^2 \, dr}_{=\frac{1}{3}} = 0.$$

Parametrize Ω:

$0 \le r \le 1, \; 0 \le \theta \le 2\pi$

<u>Q</u>: Is $\vec{F}$ conservative? <u>A</u>: <u>No</u>! $\vec{F}$ is not conservative.

EX: Use Green's Thm to compute the following line integral:



$$\int_\gamma \underbrace{\left(y+e^{\sqrt{x}}\right)}_{M} dx + \left(\underbrace{2x+\cos(y^2)}_{N}\right) dy$$

$$= \int_\gamma \vec{F}\, d\gamma \qquad \vec{F} = (M,N)$$

Green's Thm

$$\overset{\downarrow}{=} \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$\frac{\partial N}{\partial x} = 2, \; \frac{\partial M}{\partial y} = 1$

$$= \iint_\Omega 1\, dA = \text{Area}(\Omega)$$

Parametrize Ω:
$0 \le x \le 1$
$x^2 \le y \le \sqrt{x}$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx = \int_0^1 \sqrt{x} - x^2\, dx$$

$$= \left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$