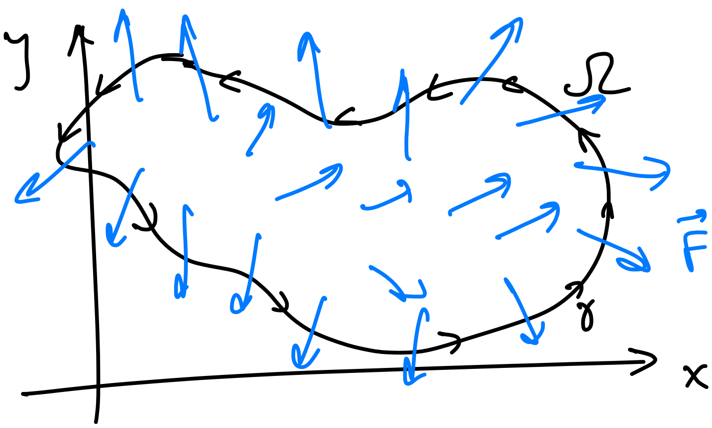
Generalizations of Green's Theorem

Recall Green's Theorem: $\Omega \subset \mathbb{R}^2$ simply-connected

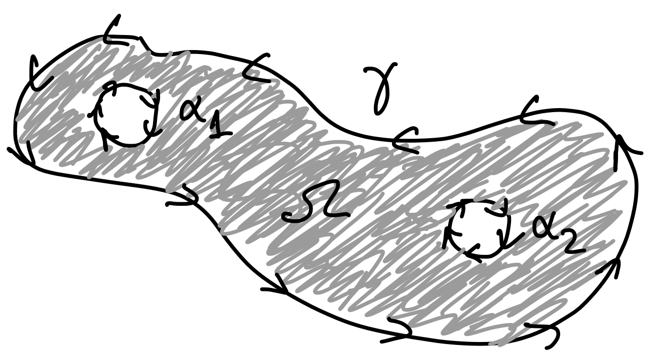
$\vec{F} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ vector field

$\vec{F} = (M, N) = M\hat{\imath} + N\hat{\jmath}$



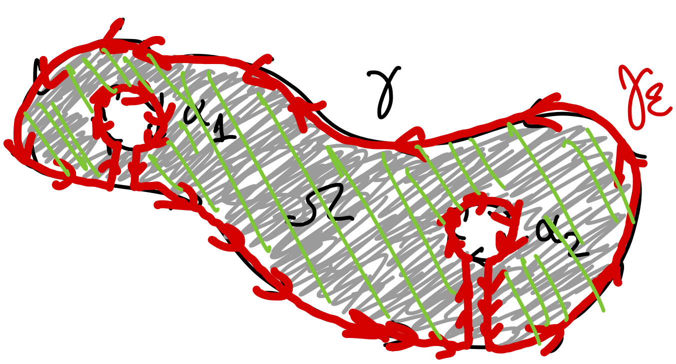
$$\boxed{\int_\gamma \vec{F}\, d\gamma = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA}$$

What if Ω is not simply-connected?



γ: outer boundary curve (counterclockwise)

α_i: inner boundary curves (clockwise)



By Green's Theorem applied to γ_ε and Ω_ε:

$$\int_{\gamma_\varepsilon} \vec{F}\, d\gamma_\varepsilon = \iint_{\Omega_\varepsilon} \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$

Note: γ_ε bounds a simply connected region Ω_ε

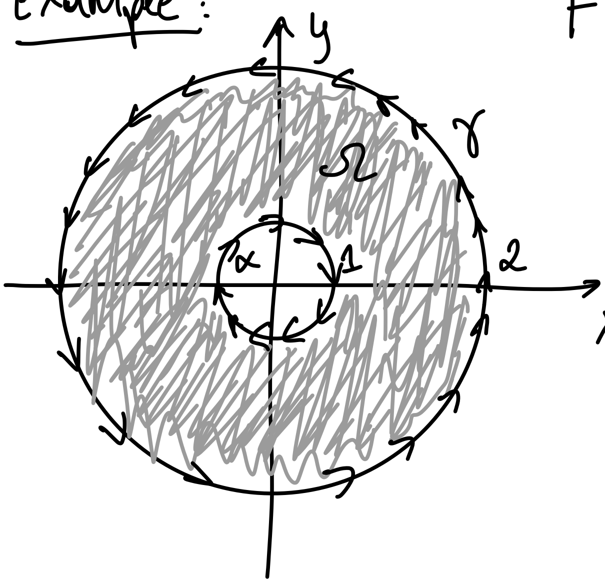
As $\varepsilon \downarrow 0$, taking limits of the above, we get

$$\int_\gamma \vec{F}\,d\gamma + \underbrace{\int_{\alpha_1} \vec{F}\,d\alpha_1 + \int_{\alpha_2} \vec{F}\,d\alpha_2}_{} = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

or as many as the interior boundary curves α_i, i.e.,

$$\sum_i \int_{\alpha_i} \vec{F}\,d\alpha_i$$

Example:



$$\vec{F}(x,y) = (x+y,\ 1) = (M,N)$$

$$\int_\gamma \vec{F}\,d\gamma + \int_\alpha \vec{F}\,d\alpha = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Compute RHS:

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -1$$

$$\iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_\Omega -1\,dA = -\iint_\Omega 1\,dA = -\text{Area}(\Omega)$$

$$= -(4\pi - \pi) = \boxed{-3\pi}$$

Compute LHS:

$$\gamma(t) = (2\cos t,\ 2\sin t), \quad t \in [0, 2\pi]$$

$$-\alpha(t) = (\cos t,\ \sin t), \quad t \in [0, 2\pi]$$ ← This is still counterclockwise!

$$\int_\gamma \vec{F}\,d\gamma = \int_0^{2\pi} \left\langle \left(\underbrace{2\cos t + 2\sin t,}_{\vec{F}(\gamma(t))}\ 1\right), \underbrace{(-2\sin t, 2\cos t)}_{\gamma'(t)} \right\rangle dt$$

$$= \int_0^{2\pi} -4\cos t\sin t - 4\sin^2 t + 2\cos t \, dt$$

$$= \int_0^{2\pi} \underbrace{-2\sin 2t + 2\cos t}_{\text{this part integrates to zero.}} - 4\sin^2 t \, dt$$

$$= -4\int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt = -2(2\pi) = -4\pi$$

$$\int_{-\alpha} \vec{F}\,d\alpha = \int_0^{2\pi} \left\langle \left(\underbrace{\cos t + \sin t,}_{\vec{F}(\alpha(t))}\ 1\right), \underbrace{(-\sin t, \cos t)}_{\alpha'(t)} \right\rangle dt$$

$$= \int_0^{2\pi} \underbrace{-\cos t\sin t}_{} - \sin^2 t + \underbrace{\cos t}_{} \, dt$$

integrate to zero

$$= -\int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt = -\pi$$

Thus $\int_\alpha \vec{F}\,d\alpha = \pi$.

Altogether:

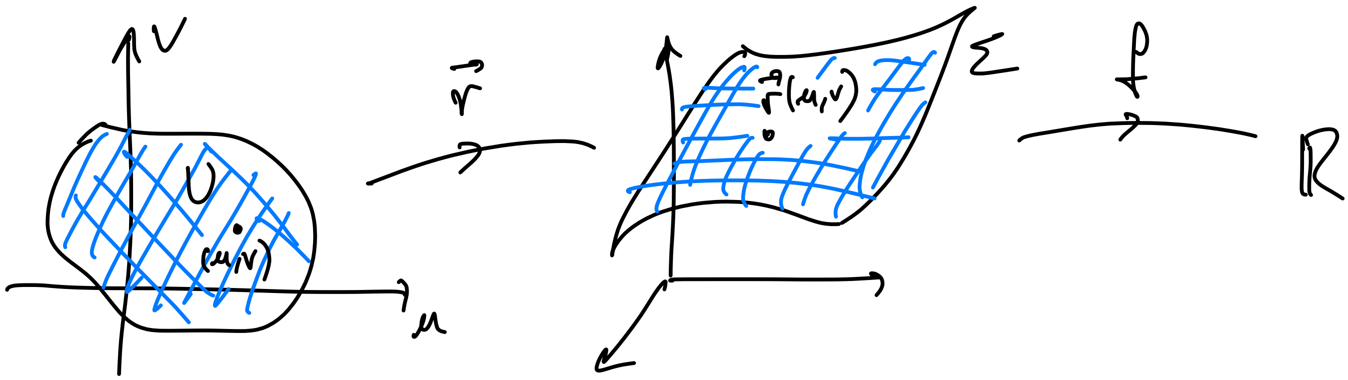
$$\int_\gamma \vec{F}\,d\gamma + \int_\alpha \vec{F}\,d\alpha = -4\pi + \pi = \boxed{-3\pi}$$

A quick tour of more advanced Vector Calculus:

(Don't worry: there will be no questions on the final exam about this)

Surface integrals:

$$\vec{r} : U \subset \mathbb{R}^2 \longrightarrow \Sigma \subset \mathbb{R}^3$$



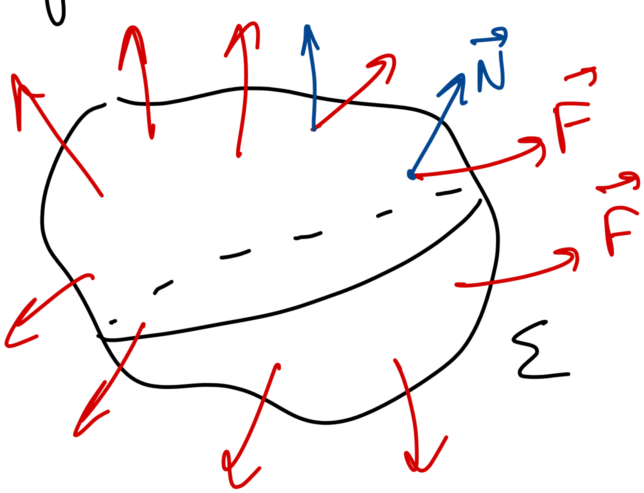
$$\vec{r}(u,v) = \big(x(u,v),\, y(u,v),\, z(u,v)\big)$$

Surface integral

$$\iint_{\Sigma} f \, dS = \iint_{U} f(\vec{r}(u,v)) \cdot \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dA$$

Flux integral:

$$\iint_{\Sigma} \langle \vec{F}, \vec{N} \rangle \, dS = \text{flux of } \vec{F} \text{ across } \Sigma.$$



$\vec{F}$ vector field

$\vec{N}$ unit vector field

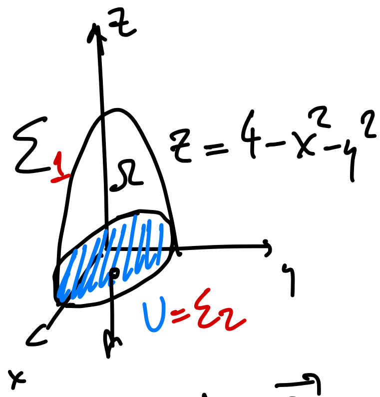
Gauss Theorem / Divergence Theorem:

$$\underbrace{\iint_{\Sigma} \langle \vec{F}, \vec{N} \rangle \, dS}_{\text{flux of } \vec{F} \text{ across } \Sigma} = \underbrace{\iiint_{\Omega} \operatorname{div} \vec{F} \, dV}_{\text{triple integral of div } \vec{F} \text{ inside the domain } \Omega \text{ bounded by } \Sigma.}$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (M, N, P)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

__Example.__ Compute the flux of $\vec{F}(x,y,z) = (2x, 2y, z)$ across the surface Σ in the picture.



$$\iint_{\Sigma = \Sigma_1 \cup \Sigma_2} \langle \vec{F}, \vec{N} \rangle \, dS = \iiint_{\Omega} \operatorname{div} \vec{F} \, dV = \iiint_{\Omega} 5 \, dV = \dots$$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(z) = 2 + 2 + 1 = 5.$$

Parametrize Ω:

$$0 \le r \le 2$$
$$0 \le \theta \le 2\pi$$
$$0 \le z \le 4 - r^2$$

$$\dots = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 5 \, dz \, r \, dr \, d\theta =$$

$$= 10\pi \int_0^2 (zr)\Big|_0^{4-r^2} dr = 10\pi \int_0^2 (4-r^2)\, r\, dr$$

$$= 10\pi \int_0^2 4r - r^3\, dr = 10\pi \left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2 = 10\pi(8-4) = \boxed{40\pi}$$

Note: We should also account for the "bottom" part of the boundary of Ω, denoted Σ_2 above:

$$\iint_{\Sigma_2} \langle \vec{F}, \vec{N} \rangle\, dS = \iint_U \underbrace{\langle (2r\cos\theta, 2r\sin\theta, 0), (0,0,-1) \rangle}_{=0}\, dA$$

$\vec{N} = (0,0,-1)$

$\vec{F}(x,y,z) = (2x, 2y, z)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 0 \end{cases} \qquad \begin{array}{l} r \in [0,2] \\ \theta \in [0,2\pi] \end{array}$$

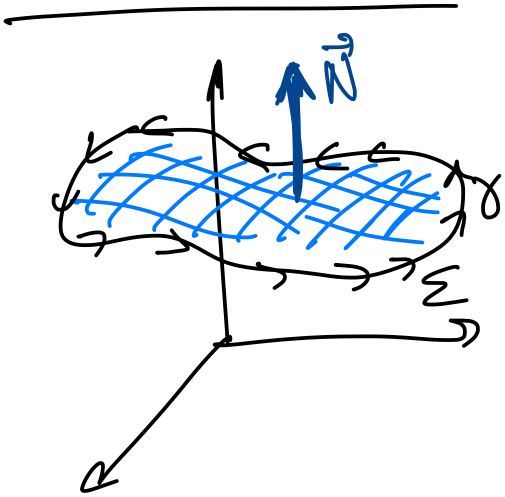
$$= \iint_U 0\, dA = \boxed{0.}$$

There's no flux through the bottom part Σ_2!

Stokes Theorem:

$\Sigma \subset \mathbb{R}^3$ surface with unit normal $\vec{N}$ and boundary γ

$\vec{F}$ vector field



$$\int_\gamma \vec{F}\, d\gamma = \underbrace{\iint_\Sigma \langle \nabla \times \vec{F}, \vec{N} \rangle\, dS}_{\text{flux of curl } \vec{F} = \nabla\times\vec{F} \text{ across } \Sigma}$$

General Version of Stokes Theorem:

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

$\Omega \subset \mathbb{R}^n$ open subset

ω is an $(n-1)$-form.