Homework Set 3

DUE: OCT 7, 2020 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. Decide if each of the statements below is **true** or **false**. If it is true, give a complete **proof**; if it is false, give an explicit **counter-example**.

   (a) If $\{x_n\}$ is a convergent sequence of real (or complex) numbers, then $\{|x_n|\}$ is also convergent.

   (b) If $\{|x_n|\}$ is a convergent sequence of real (or complex) numbers, then $\{x_n\}$ is also convergent.

   (c) If $\{x_n\}$ is a sequence of real (or complex) numbers that converges to 0, and $\{y_n\}$ is a sequence of real numbers that diverges to $+\infty$, then the sequence $\{x_n\, y_n\}$ converges to 1.

   (d) If $\{x_n\}$ is a sequence of real numbers that diverges to $+\infty$ and $a \in \mathbb{R}$, then

   $$\lim_{n\to+\infty} \sqrt{\log(x_n + a)} - \sqrt{\log x_n} = 0$$

2. Suppose $\{x_n\}$ is a Cauchy sequence in a metric space (X, d), with a subsequence $\{x_{n_k}\}$ that converges to $x \in X$, i.e., x is a subsequential limit of $\{x_n\}$. Prove that $\{x_n\}$ converges to x.

3. Given $a > 0$, define a sequence $\{x_n\}$ of real numbers inductively by setting $x_1 = \dfrac{1}{a}$, and $x_{n+1} = \dfrac{1}{a + x_n}$, i.e.,

   $$x_n = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \dots}}}.$$

   (a) Is $\{x_n\}$ monotonic?

   (b) Prove that $\{x_n\}$ converges to the unique real number L such that $L = \dfrac{1}{a+L}$, i.e., the positive root of the equation $x^2 + ax - 1 = 0$.

   Side note: Setting $a = 1$ in the above, the limit of the corresponding sequence $\{x_n\}$ is $L = \frac{1}{\varphi}$, where $\varphi = \frac{1+\sqrt{5}}{2} \cong 1.618\dots$ is the so-called *golden ratio*.