Solutions to Homework Set 3

1. Decide if each of the statements below is **true** or **false**. If it is true, give a complete **proof**; if it is false, give an explicit **counter-example**.

   (a) If $\{x_n\}$ is a convergent sequence of real (or complex) numbers, then $\{|x_n|\}$ is also convergent.

   (b) If $\{|x_n|\}$ is a convergent sequence of real (or complex) numbers, then $\{x_n\}$ is also convergent.

   (c) If $\{x_n\}$ is a sequence of real (or complex) numbers that converges to 0, and $\{y_n\}$ is a sequence of real numbers that diverges to $+\infty$, then the sequence $\{x_n\, y_n\}$ converges to 1.

   (d) If $\{x_n\}$ is a sequence of real numbers that diverges to $+\infty$ and $a \in \mathbb{R}$, then

$$\lim_{n\to+\infty} \sqrt{\log(x_n+a)} - \sqrt{\log x_n} = 0$$

Solution:

(a) **TRUE:** *First, for all $a, b \in \mathbb{C}$, using the triangle inequality, we have that:*

$$|a| = |a + (b - b)| = |(a - b) + b| \leq |a - b| + |b| \quad \Longrightarrow |a| - |b| \leq |a - b|$$

and, similarly,

$$|b| = |b + (a - a)| = |-(a - b) + a| \leq |a - b| + |a| \quad \Longrightarrow |b| - |a| \leq |a - b|$$

so, altogether, we have $\pm(|a| - |b|) \leq |a - b|$, that is, $\big||a| - |b|\big| \leq |a - b|$ for all $a, b \in \mathbb{C}$. Now, suppose $\{x_n\}$ converges to x, i.e., for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$. Applying the inequality $\big||a| - |b|\big| \leq |a - b|$ proved above, we have:

$$\big||x_n| - |x|\big| \leq |x_n - x| < \varepsilon,$$

for all $n \geq N$, that is, $\{|x_n|\}$ converges to $|x|$. $\square$

(b) **FALSE:** *Take $x_n = (-1)^n$, so that $|x_n| = 1$. Then $\{|x_n|\}$ converges to 1, but x_n does not converge.*

(c) **FALSE:** *Take $x_n = \frac{2}{n}$ and $y_n = n$. Then $\{x_n\}$ converges to 0, $\{y_n\}$ diverges to $+\infty$, but $\{x_n y_n\}$ does not converge to 1.*

(d) **TRUE:** *Using the fact that $(\sqrt{A} - \sqrt{B})(\sqrt{A} + \sqrt{B}) = A - B$, we have:*

$$\sqrt{\log(x_n + a)} - \sqrt{\log x_n} = \frac{\log(x_n + a) - \log x_n}{\sqrt{\log(x_n + a)} + \sqrt{\log x_n}} = \frac{\log\left(\dfrac{x_n + a}{x_n}\right)}{\sqrt{\log(x_n + a)} + \sqrt{\log x_n}}$$

Since $\{x_n\}$ diverges to $+\infty$, we have that $\left\{\dfrac{x_n + a}{x_n}\right\}$ converges to 1, so $\left\{\log\left(\dfrac{x_n + a}{x_n}\right)\right\}$ converges to 0. Moreover, the denominators $\left\{\sqrt{\log(x_n + a)} + \sqrt{\log x_n}\right\}$ diverge to $+\infty$. Thus, the sequence $\left\{\sqrt{\log(x_n + a)} - \sqrt{\log x_n}\right\}$ converge to 0. $\square$

2. Suppose $\{x_n\}$ is a Cauchy sequence in a metric space (X, d), with a subsequence $\{x_{n_k}\}$ that converges to $x \in X$, i.e., x is a subsequential limit of $\{x_n\}$. Prove that $\{x_n\}$ converges to x.

   Solution:

   Since the subsequence $\{x_{n_k}\}$ converges to $x \in X$, we know tht for all $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that if $n_k \geq N_1$ then $d(x_{n_k}, x) < \frac{\varepsilon}{2}$. Moreover, since $\{x_n\}$ is Cauchy, for all $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that if $n, m \geq N_2$, then $d(x_n, x_m) < \frac{\varepsilon}{2}$. Choose $\ell \in \mathbb{N}$ such that $n_\ell > \max\{N_1, N_2\}$. Then, if $n \geq \max\{N_1, N_2\}$, we have that

$$d(x_n, x) \leq d(x_n, x_{n_\ell}) + d(x_{n_\ell}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

   that is, x_n converges to x.

3. Given $a > 0$, define a sequence $\{x_n\}$ of real numbers inductively by setting $x_1 = \dfrac{1}{a}$, and $x_{n+1} = \dfrac{1}{a + x_n}$, i.e.,

$$x_n = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \dots}}}.$$

   (a) Is $\{x_n\}$ monotonic?

   (b) Prove that $\{x_n\}$ converges to the unique real number L such that $L = \dfrac{1}{a + L}$, i.e., the positive root of the equation $x^2 + ax - 1 = 0$.

   Side note: Setting $a = 1$ in the above, the limit of the corresponding sequence $\{x_n\}$ is $L = \frac{1}{\varphi}$, where $\varphi = \frac{1+\sqrt{5}}{2} \cong 1.618\ldots$ is the so-called *golden ratio.*

   Solution:

   (a) *No. The sequence $\{x_n\}$ is* not *monotonic. In fact, for all $n \in \mathbb{N}$, we have that:*

$$x_2 < x_4 < \cdots < x_{2n} < \cdots < L < \cdots < x_{2n-1} < \cdots < x_3 < x_1,$$

   i.e., the subsequence $\{x_{2n}\}$ is monotonically increasing, the subsequence $\{x_{2n-1}\}$ is monotonically decreasing, and $x_{2n} < L < x_{2n-1}$ for all $n \in \mathbb{N}$.

   Proof. *Let us first prove that the subsequence $\{x_{2n-1}\}$ is monotonically decreasing, by induction on n. The base case $n = 1$, i.e., $x_3 < x_1$ follows from:*

$$x_3 = \frac{1}{a + \frac{1}{a + \frac{1}{a}}} < \frac{1}{a + 0} = x_1.$$

Now, assume by induction that $x_{2n-1} < x_{2n-3}$. *Then*

$$x_{2n+1} = \frac{1}{a + x_{2n}} = \frac{1}{a + \frac{1}{a+x_{2n-1}}} \implies \frac{1}{x_{2n+1}} = a + \frac{1}{a + x_{2n-1}} > a + \frac{1}{a + x_{2n-3}}$$
$$\implies x_{2n+1} < \frac{1}{a + \frac{1}{a+x_{2n-3}}} = x_{2n-1},$$

that is, $x_{2n+1} < x_{2n-1}$, *which is the next case. This establishes the claim for all* $n \in \mathbb{N}$.

Similarly, we prove by induction that the subsequence $\{x_{2n}\}$ *is monotonically increasing. The base case* $n = 1$, *i.e.,* $x_2 < x_4$ *follows from:*

$$\frac{1}{x_2} = a + \frac{1}{a} = a + \frac{1}{a + 0} > a + \frac{1}{a + \frac{1}{a+\frac{1}{a}}} = \frac{1}{x_4}.$$

Now, assume by induction that $x_{2n-2} < x_{2n}$. *Then*

$$x_{2n+2} = \frac{1}{a + x_{2n+1}} = \frac{1}{a + \frac{1}{a+x_{2n}}} \implies \frac{1}{x_{2n+2}} = a + \frac{1}{a + x_{2n}} < a + \frac{1}{a + x_{2n-2}}$$
$$\implies x_{2n+2} > \frac{1}{a + \frac{1}{a+x_{2n-2}}} = x_{2n},$$

that is, $x_{2n} < x_{2n+2}$, *which is the next case. This establishes the claim for all* $n \in \mathbb{N}$.

Finally, let us show that $x_{2n} < L < x_{2n-1}$ *for all* $n \in \mathbb{N}$, *also by induction. The base case follows from:*

$$L = \frac{1}{a + L} < \frac{1}{a} = x_1$$

and, using the above inequality,

$$x_2 = \frac{1}{a + x_1} < \frac{1}{a + L} = L.$$

Now, assume by induction that $x_{2n} < L < x_{2n-1}$. *Then*

$$\frac{1}{x_{2n+1}} = a + \frac{1}{a + x_{2n-1}} < a + \frac{1}{a + L} = a + L \implies L = \frac{1}{a + L} < x_{2n+1}$$

and, similarly,

$$\frac{1}{x_{2n+2}} = a + \frac{1}{a + x_{2n}} > a + \frac{1}{a + L} = a + L \implies x_{2n+2} < \frac{1}{a + L} = L,$$

that is, $x_{2n+2} < L < x_{2n+1}$, *which is the next case. This concludes the proof.* $\square$

(b) *By what we showed in* (a), *since* $\{x_{2n}\}$ *is monotonically increasing and bounded from above by* L, *it follows that* $\{x_{2n}\}$ *is convergent, say* $x_{2n} \to \underline{L}$, *with* $\underline{L} \leq L$. *Similarly, since* $\{x_{2n-1}\}$ *is monotonically decreasing and bounded from below by* L, *it is convergent, say* $x_{2n-1} \to \overline{L}$, *with* $\overline{L} \geq L$. *We claim that* $\underline{L} = L = \overline{L}$.

Letting $n \to +\infty$ in the equations

$$x_{2n+1} = \frac{1}{a + \frac{1}{a+x_{2n-1}}}, \qquad \textit{and} \qquad x_{2n+2} = \frac{1}{a + \frac{1}{a+x_{2n}}},$$

we find that both $\underline{L}$ and $\overline{L}$ are solutions of the equation

$$z = \frac{1}{a + \frac{1}{a+\frac{1}{z}}}.$$

The above equation is equivalent to $z^2 + az - 1 = 0$, which has a unique positive real solution $z = L$. Therefore, $\underline{L} = L = \overline{L}$, and hence $\{x_n\}$ also converges to L because $\{x_n : n \in \mathbb{N}\} = \{x_{2n} : n \in \mathbb{N}\} \cup \{x_{2n-1} : n \in \mathbb{N}\}$ and L is its unique limit point.