Solutions to Homework Set 5

1. Decide if each of the statements below is **true** or **false**. If it is true, give a complete **proof**; if it is false, give an explicit **counter-example**.
(Recall that a function $f\colon X \to Y$ is called *surjective*, or *onto*, if every point of Y belongs to its image, that is, if $f(X) = Y$.)

   (a) There exists a continuous function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$.

   (b) There exists a continuous surjective function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$.

   (c) There exists a continuous function $f\colon [0,1] \to \mathbb{R}$.

   (d) There exists a continuous surjective function $f\colon [0,1] \to \mathbb{R}$.

   (e) The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, is uniformly continuous.

   (f) The function $f\colon [0,1] \to \mathbb{R}$, $f(x) = x^2$, uniformly continuous.

   Solution:

   (a) **TRUE:** *For example, any constant function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous; e.g., $f(x) = 0$ for all $x \in \mathbb{R} \setminus \{0\}$.*

   (b) **TRUE:** *For example, consider the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} x, & \text{if } x < 0, \\ x - 1, & \text{if } x > 0. \end{cases}$$

   Since the domain of f is $\mathbb{R} \setminus \{0\}$, it is continuous on its domain. Moreover, f is clearly surjective (draw its graph).

   (c) **TRUE:** *For example, any constant function $f\colon [0,1] \to \mathbb{R}$ is continuous; e.g., $f(x) = 0$ for all $x \in [0,1]$.*

   (d) **FALSE:** *Suppose, by contradiction, that such a function $f\colon [0,1] \to \mathbb{R}$ exists. Since $[0,1]$ is compact and $f\colon [0,1] \to \mathbb{R}$ is continuous, its image $f([0,1])$ is compact. Since f is surjective, $f([0,1]) = \mathbb{R}$, hence $\mathbb{R}$ is compact, which is a contradiction.*

   (e) **FALSE:** *If $f(x) = x^2$ was uniformly continuous on $\mathbb{R}$, then for all $\varepsilon > 0$ there would exist $\delta > 0$ such that $|x - y| < \delta$ implies $|x^2 - y^2| < \varepsilon$. Take $\varepsilon = 1$, and $y = x + \delta/2$. Then, given any $\delta > 0$,*

$$\left|x^2 - y^2\right| = \left|x^2 - \left(x + \frac{\delta}{2}\right)^2\right| = \left|-x\delta - \frac{\delta^2}{4}\right| = \left|x\delta + \frac{\delta^2}{4}\right| < 1$$

   cannot hold if x is chosen large enough, e.g., $x > \dfrac{1}{\delta}$.

   (f) **TRUE:** *As proven in Video 5 of Lecture 15, a continuous function on a compact set is uniformly continuous. Since $f(x) = x^2$ is continuous on the compact set $[0,1]$, it follows that $f\colon [0,1] \to \mathbb{R}$ given by $f(x) = x^2$ is uniformly continuous.*

2. Prove that a map $f\colon X \to Y$ between metric spaces is continuous if and only if the preimage $f^{-1}(C) \subset X$ of every closed subset $C \subset Y$ is closed in X.

   Solution:

   Recall that in Video 6 of Lecture 14 we proved that $f\colon X \to Y$ is continuous if and only if the preimage $f^{-1}(V)$ of every open subset $V \subset Y$ is open in X.

   Suppose $f\colon X \to Y$ is continuous, and let $C \subset Y$ be closed. Then $V = Y \setminus C$ is open, and hence $f^{-1}(V) \subset X$ is open. Since $f^{-1}(V) = X \setminus f^{-1}(C)$, it follows that $f^{-1}(C)$ is closed in X. Conversely, suppose the preimage $f^{-1}(C)$ of every closed subset $C \subset Y$ is closed in X. Let $V \subset Y$ be an open subset, so that $C = Y \setminus V$ is closed in Y. Then $f^{-1}(C) = X \setminus f^{-1}(V)$ is closed in X, hence $f^{-1}(V)$ is open in X. Thus, by the result mentioned above, $f\colon X \to Y$ is continuous. □

3. Give a rigorous proof that the equation below has at least one real solution $x \in \mathbb{R}$.

$$x^{2020} + \frac{\pi}{1 + x^2 + \sin^2 x} = 10^{100}$$

   Please do not attempt to find the solution *explicitly*.

   Solution:

   Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = x^{2020} + \frac{\pi}{1 + x^2 + \sin^2 x}.$$

   Note that $f(0) = \pi$ and $f(x) \geq x^{2020}$ for all $x \in \mathbb{R}$. Thus, given any $M > 0$, there exists $N > 0$ such that $f(x) \geq M$ for all $x \geq N$, e.g., take $N = M^{1/2020}$. In particular, setting $M = 10^{100} + 1$, there exists $N > 0$ such that $f(x) \geq 10^{100} + 1$ for all $x \geq N$. Since f is a composition of continuous functions, f is itself continuous. Therefore, by the Intermediate Value Theorem (Video 7 of Lecture 15), since $c = 10^{100}$ satisfies $f(0) \leq c \leq f(N)$, there exists $x \in [0, N]$ such that $f(x) = c$, i.e., the given equation admits at least one real solution $x \in \mathbb{R}$.