Homework Set 6

DUE: NOV 18, 2020 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. Decide if each of the statements below is **true** or **false**. If it is true, give a complete **proof**; if it is false, give an explicit **counter-example**.

   (a) There exists a monotonic function $f\colon [0,1] \to \mathbb{R}$ which is discontinuous at every point of the Cantor set P described in Lecture 6.

   (b) There exists a monotonic function $f\colon [0,1] \to \mathbb{R}$ which is continuous at every point of the Cantor set P described in Lecture 6.

   (c) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R} \setminus \{0\}$, and $\lim_{x\to 0} f'(x) = 2020$. Then $f(x)$ is also differentiable at $x = 0$ and $f'(0) = 2020$.

   (d) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous everywhere, differentiable on $\mathbb{R} \setminus \{0\}$, and $\lim_{x\to 0} f'(x) = 2020$. Then $f(x)$ is also differentiable at $x = 0$ and $f'(0) = 2020$.

   (e) The function $\psi\colon [0,1] \to \mathbb{R}$ given by

   $$\psi(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ -1, & \text{if } x \notin \mathbb{Q}, \end{cases}$$

   is Riemann-Stieltjes integrable on $[0,1]$, i.e., $\psi \in \mathcal{R}(\alpha)$.

   (f) If a bounded function $f\colon [0,1] \to \mathbb{R}$ is such that f^2 is Riemann-Stieltjes integrable, then so is f; i.e., $f^2 \in \mathcal{R}(\alpha)$ implies $f \in \mathcal{R}(\alpha)$.

2. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ satisfies

   $$|f(x) - f(y)| \leq |x - y|\, \phi(|x - y|),$$

   where $\phi\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\phi(0) = 0$. Prove that f must be a constant function.

   Hint: Compute $f'(x)$ using the definition.

3. Compute the Riemann-Stieltjes integral $\displaystyle\int_0^1 x^2\, \mathrm{d}\alpha$, where $\alpha(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{2}, \\ 5, & \text{if } x > \frac{1}{2}. \end{cases}$