Problem 1 (40 pts): Decide if each of the statements below is **true** or **false**. If it is true, give a complete **proof**; if it is false, give an explicit **counter-example**

a) **(5 pts)** The set $A = \{a + b\sqrt{2} + c\pi : a, b, c \in \mathbb{Q}\}$ is countable.

TRUE: *There is a bijection* $f\colon A \to \mathbb{Q}^3$, $f(a + b\sqrt{2} + c\pi) = (a, b, c)$, *and* $\mathbb{Q}^3$ *is countable since it is a finite Cartesian product of the countable set* $\mathbb{Q}$

b) **(5 pts)** The set $B = \bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$ is compact.

TRUE: $B = [0, 1]$ *is compact, by the Heine–Borel Theorem.*

c) **(5 pts)** If a subset $E \subset \mathbb{R}$ is such that $\forall M > 0$ there exists $x \in E$ with $|x| \geq M$, then $\sup E$ does not exist.

FALSE: *Let* $E = (-\infty, 0]$. *Clearly,* $\forall M > 0$, $x = -M \in E$ *and* $|x| = M$, *but* $\sup E = 0$.

d) **(5 pts)** $\inf_{y\in\mathbb{R}} \left(\sup_{x\in\mathbb{R}} \frac{x^2}{x^2 + y^2 + 1} \right) = \sup_{x\in\mathbb{R}} \left(\inf_{y\in\mathbb{R}} \frac{x^2}{x^2 + y^2 + 1} \right)$

FALSE: $\inf_{y\in\mathbb{R}} \underbrace{\left(\sup_{x\in\mathbb{R}} \frac{x^2}{x^2 + y^2 + 1} \right)}_{=1,\ \forall y\in\mathbb{R}} = 1$, *while* $\sup_{x\in\mathbb{R}} \underbrace{\left(\inf_{y\in\mathbb{R}} \frac{x^2}{x^2 + y^2 + 1} \right)}_{=0,\ \forall x\in\mathbb{R}} = 0$.

e) **(5 pts)** If $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$, then the sequence $\{\sin(x_n)\}$ is also Cauchy.

TRUE: *Since* $\mathbb{R}$ *is complete, a sequence is Cauchy if and only if it is convergent. Moreover,* $f(x) = \sin x$ *is continuous, so it takes convergent sequences to convergent sequences. Alternative proof: use that* $|\sin(x_n) - \sin(x_m)| \leq |x_n - x_m|$.

f) **(5 pts)** If $\{x_n\}$ is a sequence in $\mathbb{R}$ such that $\{\sin(x_n)\}$ is Cauchy, then $\{x_n\}$ is also Cauchy.

FALSE: *Take* $x_n = 2\pi n$, *and note that* $\sin(x_n) = 0$ *for all* $n \in \mathbb{N}$ *so* $\{\sin(x_n)\}$ *is clearly Cauchy, but* x_n *is not Cauchy since* $|x_n - x_m| \geq 2\pi$ *if* $n \neq m$.

g) **(5 pts)** If $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are uniformly continuous functions, then $(f \circ g)\colon \mathbb{R} \to \mathbb{R}$ is also uniformly continuous.

TRUE: *Since* f *is uniformly continuous,* $\forall \varepsilon > 0$, $\exists \delta_1 > 0$ *such that* $|x - y| < \delta_1$ *implies* $|f(x) - f(y)| < \varepsilon$. *Now, since* g *is uniformly continuous,* $\exists \delta_2 > 0$ *such that* $|t - s| < \delta_2$ *implies* $|g(t) - g(s)| < \delta_1$. *Thus, if* $t, s \in \mathbb{R}$ *satisfy* $|t - s| < \delta_2$, *then* $|f(g(t)) - f(g(s))| < \varepsilon$.

h) **(5 pts)** If $f_n\colon E \to \mathbb{R}$ is a sequence of differentiable functions that converges uniformly to $f_\infty\colon E \to \mathbb{R}$, then f_∞ is also differentiable.

FALSE: *As shown in HW7, the sequence* $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ *of differentiable functions converges uniformly on* $E = [-1, 1]$ *to* $f_\infty(x) = |x|$, *which is not differentiable at* $x = 0$.

Problem 2 (15 pts): Let $\{x_k\}$ be a convergent sequence of real numbers, with $\lim_{k\to\infty} x_k = x_\infty$. Let

$$a_n = \frac{x_1 + \cdots + x_n}{n}, \quad n \in \mathbb{N},$$

be the sequence of averages of $\{x_k\}$. Prove that $\lim_{n\to\infty} a_n = x_\infty$.

Hint: Recall $\lim_{n\to\infty} a_n = x_\infty$ means $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $|a_n - x_\infty| < \varepsilon$ if $n \geq N$.

Since x_n converges to x_∞, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x_\infty| < \varepsilon/2$ if $n \geq N$. Let $M = \max_{1\leq j\leq N-1} |x_j - x_\infty|$, and $N' \in \mathbb{N}$ be such that $N' > 2(N-1)M/\varepsilon$. Then, if $n \geq \max\{N, N'\}$, we have:

$$\begin{aligned}
|a_n - x_\infty| &= \left| \frac{x_1 + \cdots + x_n}{n} - x_\infty \right| \\
&= \left| \frac{(x_1 - x_\infty) + \cdots + (x_n - x_\infty)}{n} \right| \\
&= \left| \frac{(x_1 - x_\infty) + \cdots + (x_{N-1} - x_\infty)}{n} + \frac{(x_N - x_\infty) + \cdots + (x_n - x_\infty)}{n} \right| \\
&\leq \left| \frac{(x_1 - x_\infty) + \cdots + (x_{N-1} - x_\infty)}{n} \right| + \left| \frac{(x_N - x_\infty) + \cdots + (x_n - x_\infty)}{n} \right| \\
&\leq \frac{(N-1)}{n} \max_{1\leq j\leq N-1} |x_j - x_\infty| + \frac{|x_N - x_\infty| + \cdots + |x_n - x_\infty|}{n} \\
&< \frac{(N-1)M}{n} + \frac{(n-N+1)}{n}\frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

proving that $\lim_{n\to\infty} a_n = x_\infty$.

Problem 3 (10 pts): An isometry of a metric space (X, d) is a map $\varphi\colon X \to X$ that preserves distances, i.e., $d(\varphi(x), \varphi(y)) = d(x, y)$ for all $x, y \in X$. Suppose $f\colon X \to \mathbb{R}$ is a uniformly continuous function, and let G denote the set of all isometries of (X, d). Prove that the family $\mathcal{F} = \{(f \circ \varphi)\colon X \to \mathbb{R} : \varphi \in G\}$ is equicontinuous.

Since $f\colon X \to \mathbb{R}$ *is uniformly continuous, given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $d(x, y) < \delta$ *implies* $|f(x) - f(y)| < \varepsilon$*. Let* $\varphi \in G$ *and consider the corresponding* $(f \circ \varphi) \in \mathcal{F}$*. If* $x, y \in X$ *are such that* $d(x, y) < \delta$*, then* $d(\varphi(x), \varphi(y)) = d(x, y) < \delta$ *and therefore* $|f(\varphi(x)) - f(\varphi(y))| < \varepsilon$*. Since this holds for arbitrary* $\varphi \in G$*, it follows that the family* $\mathcal{F}$ *is equicontinuous.*

Problem 4 (15 pts): For what values of $x \in \mathbb{R}$ is the following series absolutely convergent?

$$\frac{x}{5} + \frac{x}{7} + \frac{x^2}{5^2} + \frac{x^2}{7^2} + \frac{x^3}{5^3} + \frac{x^3}{7^3} + \frac{x^4}{5^4} + \frac{x^4}{7^4} + \dots$$

Let a_n *be the nth element in the series, and note that:*

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{\dfrac{|x|^k}{5^k}} = \left(\dfrac{|x|}{5}\right)^{\frac{k}{2k-1}} & \text{if } n = 2k-1 \text{ is odd,} \\ \sqrt[n]{\dfrac{|x|^k}{7^k}} = \left(\dfrac{|x|}{7}\right)^{\frac{k}{2k}} & \text{if } n = 2k \text{ is even.} \end{cases}$$

Therefore $\limsup \sqrt[n]{|a_n|} = \lim_{k\to\infty} \left(\frac{|x|}{5}\right)^{\frac{k}{2k-1}} = \left(\frac{|x|}{5}\right)^{\frac{1}{2}} < 1$ *if and only if* $|x| < 5$*. Thus, by the Root Test, the above series is absolutely convergent if* $|x| < 5$*.*
On the other hand, if $|x| \geq 5$*, then the series diverges, since the sequence* a_n *does not converge to zero, because the subsequence* a_{2k-1} *of odd terms satisfies* $|a_{2k-1}| \geq 1$*.*

Problem 5 (20 pts): Consider the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q}, \\ \frac{1}{q^2}, & \text{if } x = \frac{p}{q} \in \mathbb{Q}, \text{ with } \gcd(p,q) = 1. \end{cases}$$

(a) Given $\varepsilon > 0$, prove that the set $F = \{x \in [0,1] : f(x) \geq \varepsilon\}$ is finite.

(b) Find a partition $P = \{0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1\}$ such that F is covered by intervals $[x_{i-1}, x_i]$ whose combined length does not exceed ε. Compute the upper and lower Riemann sums $U(f,P)$ and $L(f,P)$ with this partition.

(c) Use the above to conclude whether or not $f(x)$ is Riemann-integrable on $[0,1]$. If it is Riemann-integrable, then compute $\displaystyle\int_0^1 f(x)\,dx$.

(a) *Given $\varepsilon > 0$, we have that*

$$F = \left\{ x \in [0,1] \cap \mathbb{Q} : x = \frac{p}{q},\ \gcd(p,q) = 1,\ q^2 \leq \frac{1}{\varepsilon} \right\},$$

so there is a natural bijection between F and the set

$$\bigcup_{\substack{q \leq \frac{1}{\sqrt{\varepsilon}} \\ q \in \mathbb{N}}} \big\{ p \in \mathbb{Z} : |p| \leq q,\ \gcd(p,q) = 1 \big\}.$$

Clearly, the set of denominators $\{q \in \mathbb{N} : q \leq \frac{1}{\sqrt{\varepsilon}}\}$ is finite, and, for each such q, there are only finitely many $p \in \mathbb{Z}$ such that $|p| \leq q$ and $\gcd(p,q) = 1$. Thus, the above is a finite union of finite sets, hence finite.

(b) *Since F is finite, let us write $F = \{t_1 < t_2 < \cdots < t_k\}$, and then define*[1]

$$P = \left(\{0,1\} \cup \left\{ t_j \pm \frac{\varepsilon}{2k} : 1 \leq j \leq k \right\} \right) \cap [0,1] = \{0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1\}.$$

Up to making $\varepsilon > 0$ even smaller, we implicitly assume that $\varepsilon < k\,(t_{j+1} - t_j)$ for all $j = 1, \dots, k-1$, so that the intervals $[t_j - \frac{\varepsilon}{2k}, t_j + \frac{\varepsilon}{2k}]$ are disjoint. Clearly, the combined length of intervals $[x_{i-1}, x_i]$ that contain a point of F, i.e., intervals of the form $[t_j - \frac{\varepsilon}{2k}, t_j + \frac{\varepsilon}{2k}] \cap [0,1]$, $j = 1, \dots, k$ does not exceed ε; in other words, $\displaystyle\sum_{F \cap [x_{i-1}, x_i] \neq \emptyset} \Delta x_i \leq \varepsilon$.

Now, observe that:

- *If $F \cap [x_{i-1}, x_i] \neq \emptyset$, then $\varepsilon \leq M_i \leq 1$ and $\Delta x_i \leq \frac{\varepsilon}{k}$.*
- *If $F \cap [x_{i-1}, x_i] = \emptyset$, then $M_i < \varepsilon$.*

In both cases, $m_i = 0$ since $f(x) = 0$ on the dense set $[0,1] \setminus \mathbb{Q}$. Therefore, altogether,

$$U(f,P) = \sum_{i=1}^{n} M_i\,\Delta x_i = \sum_{F \cap [x_{i-1}, x_i] \neq \emptyset} M_i\,\Delta x_i + \sum_{F \cap [x_{i-1}, x_i] = \emptyset} M_i\,\Delta x_i$$

[1] Note that $0, 1 \in F$ if $0 < \varepsilon < 1$, so the points $t_1 - \frac{\varepsilon}{2k}$ and $t_k + \frac{\varepsilon}{2k}$ are outside the interval $[0,1]$, hence the need to intersect with $[0,1]$.

$$
\begin{aligned}
&< \sum_{F\cap[x_{i-1},x_i]\neq\emptyset} \Delta x_i \; + \sum_{F\cap[x_{i-1},x_i]=\emptyset} \varepsilon\,\Delta x_i \\
&\leq \varepsilon + \varepsilon \sum_{F\cap[x_{i-1},x_i]=\emptyset} \Delta x_i \\
&< \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}
$$

$$
L(f,P) = \sum_{i=1}^{n} m_i\,\Delta x_i = 0.
$$

(c) *By the above, for all* $\varepsilon > 0$*, there exists a partition* P *of* $[0,1]$ *such that*

$$
U(f,P) - L(f,P) < 2\varepsilon.
$$

Therefore, $f(x)$ *is Riemann-integrable on* $[0,1]$*, and*

$$
\int_0^1 f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x = 0.
$$