The number e:

$$e = \sum_{n=0}^{+\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

The above converges b/c:

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!}$$

Partial sum

$$= 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \dots + \frac{1}{1.2.3\dots n}$$

2 2 2 2 2

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}}$$

$$= 1 + \underbrace{\sum_{k=0}^{n-1} \frac{1}{2^k}}_{} < 3.$$

Partial sum of the Geom. Series

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} = \frac{1}{1 - 1/2} = 2.$$

Since $S_n < 3$, $\forall n \in \mathbb{N}$, and S_n is monotonically increasing, it follows that $\{S_n\}$ converges, i.e., $\sum_{n=0}^{+\infty} \frac{1}{n!}$ converges. (Let e denote the limit of this series).

Thm. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Pf: Denote by $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ and by $t_n = \left(1+\frac{1}{n}\right)^n$.

By the Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \quad \begin{cases} a = 1 \\ b = \frac{1}{n} \end{cases}$$

$$t_n = \left(1+\frac{1}{n}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{n^i} = \sum_{i=0}^{n} \frac{n!}{(n-i)!\, i!} \frac{1}{n^i}$$

$$= \sum_{i=0}^{n} \frac{n(n-1)(n-2)\dots(n-i+1)}{n\cdot n\cdot n \cdot\ \dots\ \cdot n} \frac{1}{i!}$$

$$= \sum_{i=0}^{n} \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-i+1}{n}\right)\frac{1}{i!}$$

$$= \sum_{i=0}^{n} \underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{i-1}{n}\right)}_{<1}\frac{1}{i!}$$

$$\left(\text{cf. } s_n = \sum_{i=0}^{n} 1\cdot\frac{1}{i!}\right) \implies t_n \le s_n, \quad \forall n \in \mathbb{N}$$

Thus $$\limsup_{n\to\infty} t_n \le \lim_{n\to\infty} s_n = e. \qquad (1)$$

If we stop at $m \le n$, we see that

$$t_n \ge 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right).$$

Fix m, let $n \to \infty$, then:

m^{th} partial sum

$$\liminf_{n\to\infty} t_n \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} = S_m$$

So we conclude that $S_m \leq \liminf_{n\to\infty} t_n$. $\forall m \in \mathbb{N}$.

Let $m \to \infty$, get:

$$e = \lim_{m\to\infty} S_m \leq \liminf_{n\to\infty} t_n. \quad (2)$$

From (1) & (2), we have

$$\limsup_{n\to\infty} t_n \leq e \leq \liminf_{n\to\infty} t_n$$

Therefore $\lim_{n\to\infty} t_n = e.$ $\square$

Speed of convergence of $\sum_{n=0}^{+\infty} \frac{1}{n!} = e$, and $e \notin \mathbb{Q}$

$$e - \overset{n\text{th partial sum}}{S_n} = e - \sum_{k=0}^{n} \frac{1}{k!}$$

$$= \sum_{k=0}^{+\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!}$$

$$= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \dots$$

$$= \frac{1}{(n+1)!}\left(1 + \frac{1}{\underbrace{n+2}_{n+1}} + \frac{1}{\underbrace{(n+2)(n+3)}_{(n+1)^2}} + \dots\right)$$

Smaller

$$\leq \frac{1}{(n+1)!}\underbrace{\left(1 + \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \dots\right)}_{\text{Geometric!}}$$

$$= \frac{1}{(n+1)!} \underbrace{\sum_{k=0}^{+\infty} \frac{1}{(n+1)^k}}_{\text{Geometric!}} = \frac{1}{(n+1)!} \, \frac{1}{\underset{\frac{n+1}{n+1}}{\underset{\|}{1}} - \frac{1}{n+1}}$$

$$= \frac{1}{(n+1)!} \, \frac{1}{\frac{n+1-1}{n+1}} = \frac{\cancel{n+1}}{\cancel{(n+1)!}} \, \frac{1}{n} = \frac{1}{n!\,n}.$$

So:

$$0 < e - s_n \overset{(*)}{<} \frac{1}{n!\,n}, \qquad \forall n \in \mathbb{N}$$

For example, $n=10$, then

$$S_{10} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{10!} \quad \text{is} \quad e - S_{10} < 10^{-7}$$

<u>Thm</u>. $e \notin \mathbb{Q}$.

<u>Pf</u>: Suppose $e \in \mathbb{Q}$; then $e = \frac{p}{q}$, $p, q \in \mathbb{N}$.

From $\circledast$; setting $n = q$:

$$0 < e - S_q < \frac{1}{q!\,q} \implies q!\left(\frac{p}{q} - S_q\right) < \frac{1}{q}$$

$$\implies \underbrace{q!\,\frac{p}{q}}_{=(q-1)!\,p \in \mathbb{N}} - \underbrace{q!\,S_q}_{\in \mathbb{N}} < \frac{1}{q} \implies$$

$\exists$ a natural <u>number</u> strictly between 0 and 1. (contradiction!).

$$q!\,S_q = q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right) = q! + q! + q(q-1)\cdots 3 + \cdots + 1 \in \mathbb{N}$$

$\square$

Root Test

if $\sqrt[n]{|a_n|}$ converges, then $\alpha = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

Thm. Given $\sum a_n$, let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

(i) $\alpha < 1 \implies \sum a_n$ converges (absolutely)

(ii) $\alpha > 1 \implies \sum a_n$ diverges

(iii) $\alpha = 1$: test is inconclusive

Pf: (i) If $\alpha < 1$, then we can choose $\alpha < \beta < 1$ and $N \in \mathbb{N}$,

$$\sqrt[n]{|a_n|} < \beta, \quad \text{for all } n \geq N$$

i.e. $|a_n| < \beta^n$. Since $0 < \beta < 1$, $\sum \beta^n$ converges

Geometric w/ ratio β

Therefore, by Comparison Thm, $\sum |a_n| < \sum \beta^n < +\infty$.

(ii) If $\alpha > 1$, then there exists a subsequence n_k s.t.

$$\sqrt[n_k]{|a_{n_k}|} \longrightarrow \alpha > 1$$

So $\sqrt[n_k]{|a_{n_k}|} > 1$ for infinitely many n_k's; i.e.

$|a_n| > 1$ for infinitely many n's.

This implies that $\sum a_n$ diverges, because it prevents $a_n \to 0$.

(iii) Consider e.g.

$$\sum_{n=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \overset{\text{(diverges)}}{=} +\infty$$

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots \overset{\text{(converges, see Lecture 10)}}{<} \infty.$$

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{\left|\frac{1}{n}\right|} = \limsup_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^2}\right|} = 1.$$

(See Lecture 10 $\sqrt[n]{n} \to 1$)

$\square$

Ratio Test

Thm. The series $\sum a_n$,

(i) converges (absolutely) if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

(ii) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$ for some fixed $n_0 \in \mathbb{N}$.

Pf: (i) if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\exists \beta$, $0<\beta<1$ and

$N \in \mathbb{N}$ s.t.

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta \quad \text{for all } n \geq N.$$

i.e. $|a_{n+1}| < \beta |a_n|$

$n=N$: $\quad |a_{N+1}| < \beta |a_N|$

$n=N+1$: $\quad |a_{N+2}| < \beta |a_{N+1}| < \beta \cdot \beta |a_N| = \beta^2 |a_N|$

$n = N+2$: $\qquad |a_{N+3}| < \beta^3 |a_N|$

$\vdots \qquad\qquad \vdots$

$n = N+p$ $\qquad |a_{\underbrace{N+p}_{n}}| < \beta^p |a_N| \qquad p = n-N$

That is, $|a_n| < \beta^{n-N} |a_N| = \beta^{-N} |a_N| \beta^n$ for all $n \geq N$.

only part that depends on n.

$$\sum |a_n| < \beta^{-N} |a_N| \sum \beta^n < \infty.$$

Geometric series w/ ratio $\beta < 1$.

(ii) If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for n suff. large, then $|a_{n+1}| \geq |a_n|$ for n suff. large, which implies that $a_n \not\to 0$. So $\sum a_n$ diverges.

$\square$

Remark: If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, nothing can be said:

$$\sum \frac{1}{n} = +\infty, \qquad \sum \frac{1}{n^2} < \infty$$

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \longrightarrow 1 \qquad \frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \longrightarrow 1.$$

Example

$$\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots$$

$$\underbrace{\frac{1}{2^1} + \frac{1}{2^0}}_{2} + \underbrace{\frac{1}{2^3} + \underset{a_n}{\frac{1}{2^2}}}_{2} + \underbrace{\underset{a_{n+1}}{\frac{1}{2^5}} + \frac{1}{2^4}}_{2} + \underbrace{\underset{a_n}{\frac{1}{2^7}} + \underset{a_{n+1}}{\frac{1}{2^6}}}_{2} + \cdots$$

Notice that this series is a rearrangement of the geometric series $\sum_{n=0}^{+\infty} \frac{1}{2^n}$

Ratio test:

$$\frac{a_{n+1}}{a_n} = \begin{cases} 2 & \text{(for ratios in green)} \\ \frac{1}{8} & \text{(for ratios in red)} \end{cases}$$

$$\underbrace{\liminf_{n\to\infty} \frac{a_{n+1}}{a_n}}_{\text{smallest subsequential limit of } \frac{a_{n+1}}{a_n}} = \frac{1}{8} \qquad \underbrace{\limsup_{n\to\infty}}_{\text{largest subseq. limit of } \frac{a_{n+1}}{a_n}} \frac{a_{n+1}}{a_n} = 2$$

Ratio test does not apply!

Root test: $\phi : \mathbb{N}_0 \to \mathbb{N}_0$

$\phi(0)=1, \ \phi(1)=0, \ \phi(2)=3, \ \phi(3)=2, \ \dots$

$\phi(n) = \underline{\qquad ? \qquad}$ (exercise)

$$\frac{1}{2}+1+\frac{1}{8}+\frac{1}{4}+\frac{1}{32}+\frac{1}{16}+\frac{1}{128}+\frac{1}{64}+\cdots=\sum_{n=0}^{+\infty}\frac{1}{2^{\phi(n)}}$$

$$\lim_{n\to\infty}\frac{\phi(n)}{n}=1. \quad \text{b/c} \quad n-1\le\phi(n)\le n+1$$

$$\Downarrow$$

$$\frac{n-1}{n}\le\frac{\phi(n)}{n}\le\frac{n+1}{n}$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$1$$

(By the Squeeze thm)

$$\sqrt[n]{|a_n|}=\sqrt[n]{\frac{1}{2^{\phi(n)}}}=2^{-\frac{\phi(n)}{n}}$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}2^{-\frac{\phi(n)}{n}}=2^{-\lim_{n\to\infty}\frac{\phi(n)}{n}}=2^{-1}=\boxed{\frac{1}{2}<1}$$

By the Root test it follows that series converges.

Example:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$$

$$\frac{1}{2^2}\cdot\frac{3}{1} \qquad \frac{1}{2^3}\cdot\frac{3^2}{1} \qquad \frac{1}{2^4}\cdot\frac{3^3}{1} \qquad \cdots \qquad \frac{1}{2^{n+1}}\cdot\frac{3^n}{1}=\frac{1}{2}\left(\frac{3}{2}\right)^n$$

$$\frac{1}{3}\cdot\frac{2}{1} \qquad \frac{1}{3^2}\cdot\frac{2^2}{1} \qquad \frac{1}{3^3}\cdot\frac{2^3}{1} \qquad \cdots \qquad \frac{1}{3^n}\cdot\frac{2^n}{1}=\left(\frac{2}{3}\right)^n$$

Ratio test:

$$\frac{a_{n+1}}{a_n}=\begin{cases}\left(\frac{2}{3}\right)^n & \text{(for ratios in blue)}\\[2ex] \frac{1}{2}\left(\frac{3}{2}\right)^n & \text{(for ratios in red)}\end{cases}$$

$$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\Big(\underbrace{\tfrac{2}{3}}_{<1}\Big)^n=0$$

$$\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{2}\Big(\underbrace{\tfrac{3}{2}}_{>1}\Big)^n=+\infty$$

The ratio test does not apply ...

Root test: $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$

3 4 5 6 7 8

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[2n]{\frac{1}{2^n}} & \text{for } a_n \text{ in blue} \\ \sqrt[2n]{\frac{1}{3^n}} & \text{for } a_n \text{ in red.} \end{cases}$$

$$\liminf_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}}$$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^n}} = \boxed{\frac{1}{\sqrt{2}} < 1}$$

Root test implies that the series converges!

<u>Note</u>: In both examples, the Root test yielded convergence of the series, while the Ratio test did not apply.

<u>Thm</u>. For any sequence $\{a_n\}$ of positive real numbers,

(1) $$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n}$$

(2) $$\limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$$

If ratio test applies to give convergence, then so does the root test.
(in general, the root test may apply w/o the ratio test applying)

<u>Pf</u>: (1) is left as an exercise (similar to (2)).

(2) $$\alpha = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$$ If $\alpha = +\infty$, then there is nothing to do.

Suppose $\alpha < +\infty$. Let $\beta > \alpha$. Let $N \in \mathbb{N}$ s.t.

$$\frac{a_{n+1}}{a_n} \leq \beta \qquad \text{for all } n \geq N.$$

For all $p \in \mathbb{N}$, if $n \geq N$:

$n = N \qquad a_{N+1} \leq \beta\, a_N$

$n = N+1 \qquad a_{N+2} \leq \beta\, a_{N+1} \leq \beta^2 a_{N+2}$

$\vdots$

$n = N+p \qquad a_{N+p} \leq \beta\, a_{N+p-1} \leq \beta^2 a_{N+p-2} \leq \dots \leq \beta^p a_N$

That is, $\quad 0 \leq a_n \leq a_N \beta^{-N} \cdot \beta^n \quad$ where $n = N+p$.

Taking n^{th} root:

$$\sqrt[n]{a_n} \le \sqrt[n]{(a_N \beta^{-N})} \cdot \beta$$

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \le \beta. \quad \underbrace{\limsup_{n\to\infty} \sqrt[n]{(\underbrace{a_N \beta^{-N}}_{\text{does not depend on } n})}}_{\to 1} = \beta.$$

Since this holds for all $\beta > \alpha = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$, we have that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

$\square$