Rearrangements

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} < \infty$$

Alt. Series Test

$$\text{"} \log 2 < \frac{5}{6}.$$

$$s = \underbrace{1 - \frac{1}{2} + \frac{1}{3}}_{\frac{1}{2}+\frac{1}{3}=\frac{5}{6}} \underbrace{- \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots}_{<0} < \frac{5}{6}$$

Rearrange the above series with two positive terms together, followed by one negative term:

$$s' = \underbrace{1 + \frac{1}{3} - \frac{1}{2}}_{k=1} + \underbrace{\frac{1}{5} + \frac{1}{7} - \frac{1}{4}}_{k=2} + \underbrace{\frac{1}{9} + \frac{1}{11} - \frac{1}{6}} + \cdots > \frac{5}{6}$$

$$\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0, \quad \forall k \in \mathbb{N}$$

So

s'_n = partial sum up to n satisfies: $s'_3 < s'_6 < s'_9 < \cdots$

$$\limsup_{n\to\infty} s'_n > s'_3 = 1 + \frac{1}{3} - \frac{1}{2} = \frac{5}{6}$$

So: changing the order of the summands changes the answer!

(of course, this wouldn't happen with finitely many terms, but, here, we have an infinite sum...)

Def: $\sum a_n'$ is a <u>reorrangement</u> of $\sum a_n$ if $a_n' = a_{n_k}$ where $\{n_k\}$ is an enumeration of $\mathbb{N}$, i.e.,

$$\begin{aligned} \mathbb{N} &\longrightarrow \mathbb{N} \\ k &\longmapsto n_k \end{aligned} \quad \text{is a bijection.}$$

Theorem (Riemann). Let $\sum a_n$ be a <u>conditionally convergent</u> series. Given any α, β s.t.

converges, but <u>not</u> absolutely

$$-\infty \le \alpha \le \beta \le +\infty$$

there exists a reorrangement $\sum a_n'$ of $\sum a_n$, such that its partial sums s_n' satisfy:

$$\liminf_{n\to\infty} s_n' = \alpha, \qquad \limsup_{n\to\infty} s_n' = \beta.$$

Cor: "You can change the order of summation in $\sum a_n$ and make it converge to whatever prescribed number you want!"

Proof: Let

$$p_n = \frac{|a_n| + a_n}{2} \overset{\text{If } a_n \ge 0}{=} a_n \qquad \text{and} \qquad q_n = \frac{|a_n| - a_n}{2} \overset{\text{If } a_n \le 0}{=} -a_n.$$

So that $p_n - q_n = a_n$, $p_n + q_n = |a_n|$, $p_n \ge 0$, $q_n \ge 0$.

Claim: $\sum p_n$ and $\sum q_n$ diverge.

Pf. If $\sum p_n < \infty$ and $\sum q_n < \infty$, then $\underbrace{\sum (p_n + q_n)}_{\substack{\| \\ \sum |a_n|}} < \infty$ contradiction.

diverges

So at least one among $\sum p_n$, $\sum q_n$ must diverge.

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (p_n - q_n) = \underbrace{\sum_{n=1}^{N} p_n - \sum_{n=1}^{N} q_n}$$

converges

if at least one of these diverges, then the entire sum diverges (contradiction)

Therefore, both must diverge.

Let $P_1, P_2, P_3, \ldots$ denote the nonnegative terms of $\sum a_n$ (in the original order!) and $Q_1, Q_2, Q_3, \ldots$ be the absolute values of the negative terms of $\sum a_n$.

Note $\sum P_n$, $\sum Q_n$ differ from $\sum p_n$, $\sum q_n$ only by zeroes, so they also diverge.

Given α, β, choose sequences $\alpha_n \to \alpha$, $\beta_n \to \beta$, with $\alpha_n < \beta_n$, $\beta_1 > 0$. We will construct sequences $\{m_n\}$ and $\{k_n\}$ s.t.

$$\underbrace{P_1 + \cdots + P_{m_1}} \underbrace{- Q_1 - \cdots - Q_{k_1}} + \underbrace{P_{m_1+1} + \cdots + P_{m_2}} \underbrace{- Q_{k_1+1} - \cdots - Q_{k_2}} + \cdots$$

(which is clearly a rearrangement of $\sum a_n$) satisfies the $\liminf = \alpha$, $\limsup = \beta$ claim.

Let m_1, k_1 be the smallest natural numbers s.t.

elements of seq. approximating α, β ("targets")

$$P_1 + \dots + P_{m_1} > \beta_1$$

$$P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} < \alpha_1$$

Let m_2, k_2 be the smallest natural numbers s.t.

$$P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} + P_{m_1+1} + \dots + P_{m_2} > \beta_2$$

$$P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} + P_{m_1+1} + \dots + P_{m_2} - Q_{k_1+1} - \dots - Q_{k_2} < \alpha_2$$

Continuing in this way, we may construct the above sequences as claimed since $\sum P_n$, $\sum Q_n$ diverge.

Denote by x_n and y_n the partial sums of

$$\underbrace{P_1 + \dots + P_{m_1}} - \underbrace{Q_1 - \dots - Q_{k_1}} + \underbrace{P_{m_1+1} + \dots + P_{m_2}} - \underbrace{Q_{k_1+1} - \dots - Q_{k_2}} + \dots$$

whose last terms are P_{m_n} and $-Q_{k_n}$, then

$$|x_n - \beta_n| \le P_{m_n}, \qquad |y_n - \alpha_n| \le Q_{k_n}$$

Since $a_n \to 0$, $P_n \to 0$, $Q_n \to 0$ as $n \to \infty$ and hence $x_n \to \beta$, $y_n \to \alpha$. This proves that there are subsequential limits (of partial sums) to α and β as desired. $\square$

<u>Thm</u>. If $\sum a_n$ converges absolutely, then <u>any</u> <u>rearrangement</u> of $\sum a_n$ also converges to the same limit.

<u>Pf</u>: Let $\sum a_n'$ be a rearrangement with partial sums s_n'. Given any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n, m \geq N$,

$$\sum_{i=n}^{m} |a_i| \leq \varepsilon$$

Choose p s.t. $1, 2, \ldots, N$ are contained in the set $\{k_1, \ldots, k_p\}$, where $a_n' = a_{k_n}$. If $n > p$, the numbers $a_1, \ldots, a_N$ cancel each other in the difference of partial sums $s_n - s_n'$. Thus,

$$|s_n - s_n'| \leq \left| \sum_{i=n}^{m} a_i \right| \leq \sum_{i=n}^{m} |a_i| \leq \varepsilon.$$

i.e., s_n' converges to same limit as s_n. $\square$

Exercises

Babylonian method to approximate square roots.

Rudin Chap 3 #16: Given $\alpha > 0$, choose $x_1 > \sqrt{\alpha}$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right); \qquad n \in \mathbb{N}$$

a) Prove $\{x_n\}$ is monotonically decreasing and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$

$$x_2 = \frac{1}{2}\Big(x_1 + \underbrace{\frac{\alpha}{x_1}}_{< x_1}\Big) < x_1 \qquad\qquad x_1^2 > \alpha \iff \frac{\alpha}{x_1} < x_1$$

average of x_1 and smth smaller than x_1

⊛ Recall: $\forall a, b \in \mathbb{R}_+$, $a \neq b$

$$\frac{a+b}{2} > \sqrt{ab}$$

Applying this to x_{n+1}:

$$x_{n+1} = \frac{1}{2}\Big(x_n + \underbrace{\frac{\alpha}{x_n}}_{\circledast\ < x_n}\Big) < x_n$$

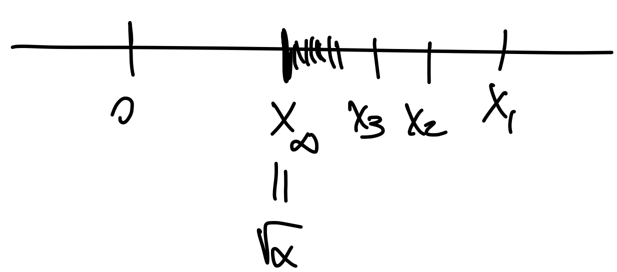
average of x_n & smth smaller than x_n.

$$x_{n+1} = \frac{x_n + \alpha/x_n}{2} > \sqrt{x_n \cdot \frac{\alpha}{x_n}} = \sqrt{\alpha}.$$

$$\implies x_{n+1}^2 > \alpha \implies \frac{\alpha}{x_{n+1}} < x_{n+1}$$

This shows that $\{x_n\}$ is monotonically decreasing. It is also, clearly, bounded from below by 0. So it must converge:

$$x_n \longrightarrow x_\infty$$



Claim: $x_\infty = \sqrt{\alpha}$.

Take $n \to \infty$ in both sides of the recurrence relation:

$$x_\infty = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = \frac{1}{2}\left(x_\infty + \frac{\alpha}{x_\infty}\right)$$

$$2x_\infty^2 = x_\infty^2 + \alpha \implies x_\infty^2 = \alpha \implies x_\infty = \sqrt{\alpha}. \quad \square$$

b) Let $\varepsilon_n = x_n - \sqrt{\alpha}$, show $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$

we know: $x_n > \sqrt{\alpha}, \ \forall n \in \mathbb{N}$ (*)

$$\varepsilon_{n+1} = x_{n+1} - \sqrt{\alpha}$$

$$\begin{aligned}
2x_n \varepsilon_{n+1} &= 2x_n(x_{n+1} - \sqrt{\alpha}) \\
&= 2x_n x_{n+1} - 2x_n\sqrt{\alpha} \\
&= \cancel{2} x_n \frac{1}{\cancel{2}}\left(x_n + \frac{\alpha}{x_n}\right) - 2x_n\sqrt{\alpha} \\
&= x_n^2 + \alpha - 2x_n\sqrt{\alpha} \\
&= (x_n - \sqrt{\alpha})^2 \\
&= \varepsilon_n^2
\end{aligned}$$

$$\implies \varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} \quad \square$$

This can be used to show that the sequence x_n converges to $\sqrt{\alpha}$ very quickly.

Setting $\beta = 2\sqrt{\alpha}$, we have:

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$$

Indeed,

$$\varepsilon_2 < \frac{\varepsilon_1^2}{2\sqrt{\alpha}} = \frac{\varepsilon_1^2}{\beta}$$

$$\varepsilon_3 < \frac{\varepsilon_2^2}{\beta} < \frac{1}{\beta}\left(\frac{\varepsilon_1^2}{\beta}\right)^2 = \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^2}$$

$$\vdots$$

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}.$$

Rudin Chap 3 # 9 ← This might be helpful in HW4 #2

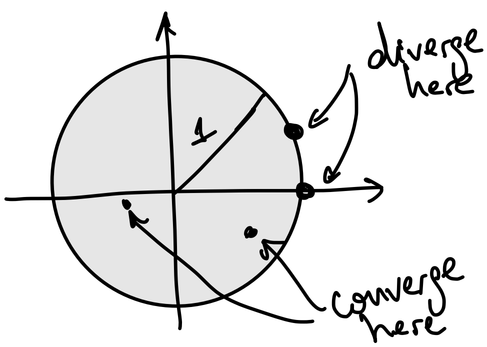
Find the radius of convergence:

a) $\sum n^3 z^n$

$$\limsup_{n\to\infty} \sqrt[n]{|n^3 z^n|} = \lim_{n\to\infty} n^{\frac{3}{n}} |z| = |z| \Big(\underbrace{\lim_{n\to\infty} n^{\frac{1}{n}}}_{=1}\Big)^3 = |z|.$$

If $|z|<1$, then we have convergence (by Root test).

If $|z|=1$, then $|n^3 z^n| = n^3 \nearrow +\infty$, so series will diverge



$R = 1.$

(b) $\sum \frac{2^n}{n!} z^n = e^{2z}$

side comment:

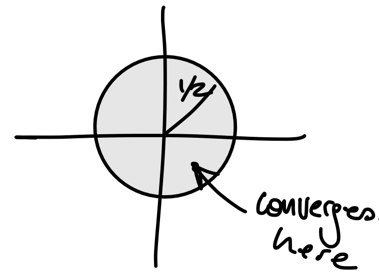
$$e^z = \sum \frac{z^n}{n!}$$

$$e^{2z} = \sum \frac{(2z)^n}{n!} = \sum \frac{2^n z^n}{n!}$$

$$\limsup_{n\to\infty} \left| \underbrace{\frac{2^{n+1} z^{n+1}}{(n+1)!}}_{a_{n+1}} \cdot \underbrace{\frac{n!}{2^n z^n}}_{1/a_n} \right| = \lim_{n\to\infty} \frac{2}{n+1}|z| = 0. < 1$$

So, by the Ratio test, the above always converges, no matter how large $|z|$ is. Thus, $R = +\infty$.

(c) $\sum \frac{2^n}{n^2} z^n$



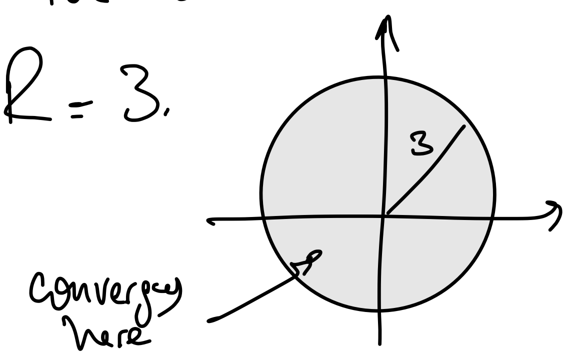
$$\limsup_{n\to\infty} \sqrt[n]{\left|\frac{2^n z^n}{n^2}\right|} = \lim_{n\to\infty} \frac{2|z|}{n^{2/n}} = \frac{2|z|}{\Big(\underbrace{\lim_{n\to\infty} n^{\frac{1}{n}}}_{=1}\Big)^2} = 2|z|.$$

If $|z| < \frac{1}{2}$, then $2|z| < 1$, so the above gives convergence by the Root test. So, $R = 1/2$.

(d) $\sum \frac{n^3}{3^n} z^n$

$$\limsup_{n\to\infty} \sqrt[n]{\left|\frac{n^3 z^n}{3^n}\right|} = \lim_{n\to\infty} \frac{n^{\frac{3}{n}}|z|}{3} = \frac{|z|}{3}\Big(\underbrace{\lim_{n\to\infty} n^{\frac{1}{n}}}_{=1}\Big)^3 = \frac{|z|}{3}.$$

If $|z| < 3$, then $\frac{|z|}{3} < 1$, so by the above we have convergence by the Root test. So $R = 3$.



$$\left(\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)$$