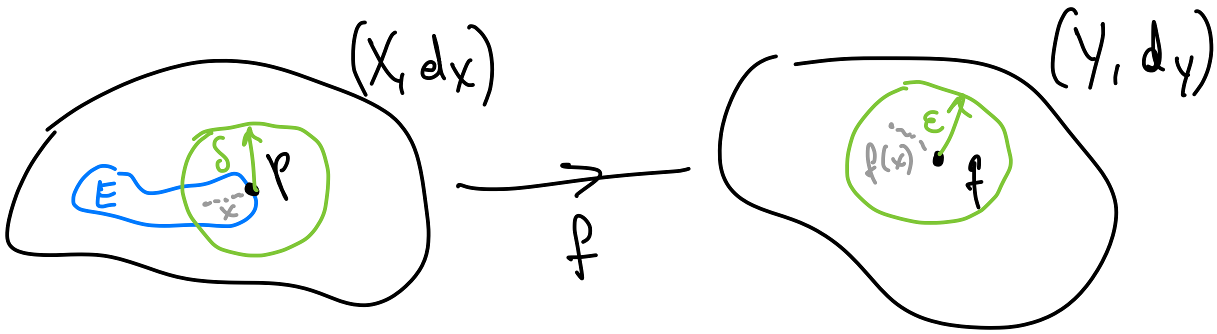
Limits of Functions

(X, d_X), (Y, d_Y) metric spaces, $f: X \to Y$

Def: Suppose $E \subset X$ is s.t. $f(E) \subset Y$ and $p \in X$ is a limit point of E. (Note it might be that $p \notin E$)
We say $\lim_{x \to p} f(x) = q$ if $q \in Y$ satisfies

$$\forall \varepsilon > 0\ \exists \delta > 0,\ \forall x \in E,\ 0 < d_X(x,p) < \delta \implies d_Y(f(x), q) < \varepsilon.$$



Remark: If $(X, d_X) = (Y, d_Y) = \mathbb{R}$ w/ usual distance $d(a,b) = |b-a|$, then the above coincides with the usual definition of limits for $f: \mathbb{R} \to \mathbb{R}$.

Thm: $\lim_{x \to p} f(x) = q \iff \lim_{n \to \infty} f(p_n) = q$ for all sequences $\{p_n\}_{n \in \mathbb{N}}$ in E s.t. $p_n \neq p$ and $p_n \to p$.

Pf: ($\Longrightarrow$) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. for all $x \in E$,

$0 < d(x,p) < \delta \Rightarrow d(f(x), q) < \varepsilon$

Given $\{p_n\}$ seq. in E s.t. $p_n \neq p$. $p_n \to p$,
there exists $N \in \mathbb{N}$ s.t. $n \geq N$ implies
$0 < d(p_n, p) < \delta$. By the above, $d(f(p_n), q) < \varepsilon$.
That is, $\lim_{n\to\infty} f(p_n) = q$. $\quad f(p_n) \longrightarrow q$.

($\Longleftarrow$) Suppose $\lim_{x\to p} f(x) \neq q$. Then $\exists \varepsilon > 0$, s.t. $\forall \delta > 0$,
$\exists x \in E$ for which $0 < d(x,p) < \delta$ but $d(f(x), q) \geq \varepsilon$.
Let $\delta_n = \frac{1}{n}$, then there exists a sequence
$\{p_n\}$ in E, $0 < d(p_n, p) < \delta_n = \frac{1}{n}$. By hypothesis,
$\lim_{n\to\infty} f(p_n) = q$. This contradicts $d(f(x), q) \geq \varepsilon$
if $x \in E$, $0 < d(x,p) < \delta_n$. So it must be $\lim_{x\to p} f(x) = q$ $\square$

Cor: If f has a limit at p, it is unique.

Pf: Recall (Lecture 8, Video 2) that limits of sequences are unique.

Thm: Suppose $f, g : X \to \mathbb{R}$ (or $\mathbb{C}$),
p is a limit point of $E \subset X$, and

$$\lim_{x \to p} f(x) = A, \quad \lim_{x \to p} g(x) = B.$$

Then:

a) $\displaystyle\lim_{x \to p} (f+g)(x) = A + B$

b) $\displaystyle\lim_{x \to p} (f \cdot g)(x) = A \cdot B$

c) $\displaystyle\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$.

Pf: Recall (Lecture 8, Video 3) that the analogous properties hold for limits of sequences.
Thus, the above claims follow by applying the previous theorem. $\square$

Example: $\displaystyle\lim_{x \to p} (f+g)(x) = A + B$

Know $\displaystyle\lim_{x \to p} f(x) = A \overset{\text{Thm}}{\Longrightarrow}$ $\forall \{p_n\}$ in E, $p_n \to p$, $p_n \neq p$. $f(p_n) \to A$.

$\displaystyle\lim_{x \to p} g(x) = B \overset{\text{Thm}}{\Longrightarrow}$ $\forall \{p_n\}$ in E, $p_n \to p$, $p_n \neq p$, $g(p_n) \to B$.

Use Thm from Lecture 8 about sequences:

$(f+g)(p_n) = f(p_n) + g(p_n) \longrightarrow A+B$

i.e. $\forall \{p_n\}$ in E, $p_n \neq p$, $p_n \to p$. $\quad (f+g)(p_n) \to A+B$

By Thm above: $\lim_{x\to p} (f+g)(x) = A+B$. $\square$

Continuous Functions

$$f: \underset{\underset{E}{\cup}}{X} \longrightarrow Y, \qquad p \in E$$

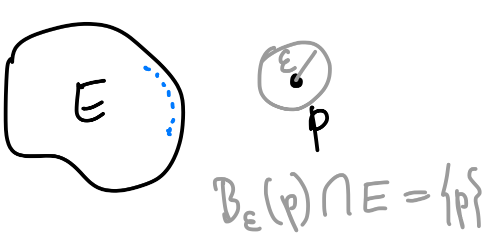
<u>Def:</u> We say f is <u>continuous at $p \in E$</u> if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

for all $x \in E$ $\qquad d_X(x,p) < \delta \implies d_Y(f(x), f(p)) < \varepsilon$.

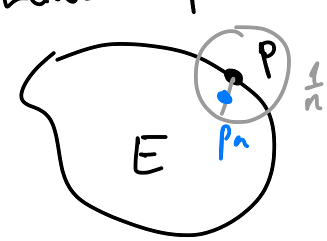
We say f is <u>continuous</u> on E if it is continuous at all points $p \in E$.

<u>Note</u>: If $p \in E$ is <u>isolated</u>, then <u>any</u> function f is continuous at p. So continuity only imposes restrictions on limit points of E.

Isolated



Limit point



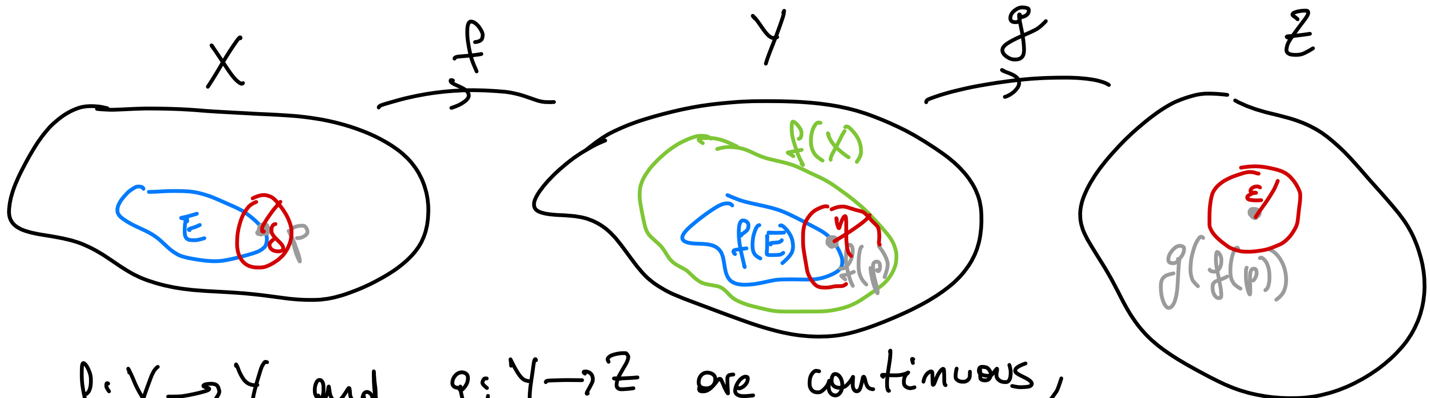
$d(p,p_n) < \frac{1}{n}$
$p_n \neq p$
$p_n \in E$

Thm: Suppose $p \in E$ is a limit point of E. Then

$$f \text{ is continuous at } p \iff \lim_{x \to p} f(x) = f(p).$$

Pf: Immediate from the above definitions (Exercise). $\square$

Thm: Composition of continuous functions is continuous.



If $f: X \to Y$ and $g: Y \to Z$ are continuous, then $(g \circ f): X \to Z$ is also continuous. ← (Also point by point...)

Pf: Given $\varepsilon > 0$, since g is cont. at $f(p) \in Y$, there exists $\eta > 0$ s.t.

$$d_Y(y, f(p)) < \eta,\ y \in f(E) \implies d_Z(g(y), g(f(p))) < \varepsilon.$$

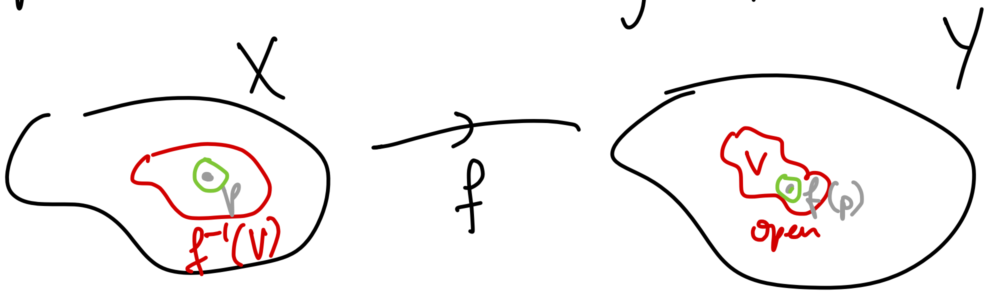
Using the fact that f is continuous at p, there exists $\delta > 0$ s.t.

$$d_X(x, p) < \delta,\ x \in E \implies d_Y(f(x), f(p)) < \eta$$

Thus, $d_X(x,p) < \delta,\ x \in E \implies d_Z(g(f(x)), g(f(p))) < \varepsilon$,

i.e., $(g \circ f)$ is continuous at $p \in X$. $\square$

Thm: $f: X \to Y$ is <u>continuous</u> if and only if $f^{-1}(V)$ is open in X for every open set $V \subset Y$



<u>Recall</u>: $f^{-1}(V) = \{x \in X : f(x) \in V\}$.

<u>Pf</u>: If f is cont. on X and $V \underset{\text{open}}{\subset} Y$, then we want to show that every $p \in f^{-1}(V)$ is an interior point of $f^{-1}(V)$. Since $p \in f^{-1}(V)$, we have $f(p) \in V$. Because V is open, there exists $\varepsilon > 0$ s.t. $y \in V$ for all $d(y, f(p)) < \varepsilon$. Since f is cont. at p, there exists $\delta > 0$ s.t. if $d_X(x,p) < \delta$, then $d_Y(f(x), f(p)) < \varepsilon$. Thus if $d_X(x,p) < \delta$, we have $f(x) \in V$, i.e., $x \in f^{-1}(V)$.

To prove the converse, suppose $f^{-1}(V) \subset X$ is open for all $V \underset{\text{open}}{\subset} Y$. Given $p \in X$, let

$$V = \{y \in Y : d_Y(y, f(p)) < \varepsilon\} = B_\varepsilon(f(p))$$

Since $V \subset Y$ is open, it follows that $f^{-1}(V) \subset X$ is open; that is, $\exists \delta > 0$ s.t. $x \in f^{-1}(V)$ if $d_X(p,x) < \delta$, i.e., if $x \in f^{-1}(V)$ and $d_X(p,x) < \delta$ then $d_Y(f(p), f(x)) < \varepsilon$. $\square$

Continuous functions with values in $\mathbb{R}$ or $\mathbb{R}^n$ ($\mathbb{C}$ or $\mathbb{C}^n$).

Thm: If $f, g : X \to \mathbb{R}$ (or $\mathbb{C}$) are continuous, then $f+g$, $f \cdot g$, and f/g are also continuous on X.

Pf: At isolated points of X, there's nothing to do. At limit points of X, use the corresponding properties for limits and the characterization of continuity in terms of limits. $\square$

Def: If $f : X \to \mathbb{R}^n$, then $f(x) = (f_1(x), \dots, f_n(x))$, where $f_i : X \to \mathbb{R}$ are called coordinate functions of f.

Thm: $f : X \to \mathbb{R}^n$ is continuous if and only if all of its coordinate functions $f_i : X \to \mathbb{R}$, $i = 1, \dots, n$ are continuous.

Pf:

$$\underbrace{|f_i(x) - f_i(y)|}_{d(f_i(x), f_i(y))} \leq \underbrace{|f(x) - f(y)|}_{d(f(x), f(y))} = \sqrt{\sum_{i=1}^{n} |f_i(x) - f_i(y)|^2}$$

If f is cont., then f_i are continuous by the above. Conversely, if f_i are cont., then f is cont. by above. $\square$

Cor: Examples of continuous functions

$f_i : \mathbb{R}^n \to \mathbb{R}$, $f_i(x) = x_i$ "ith projection", $i = 1, \dots, n$

are continuous

(since $|x_i - y_i| \leq |x - y|$, $\forall i$)

Products, sums, etc. will also be continuous.

In particular, all polynomials are continuous.