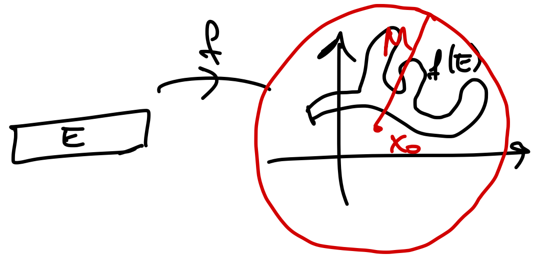
Continuity and compactness

Def: A map $f: E \to (X, d)$ is <u>bounded</u> if there exists $M \in \mathbb{R}$ s.t. $f(E) = \{x \in X : x = f(p), p \in E\}$ satisfies $f(E) \subset B_M(x_0)$ for some $x_0 \in X$; i.e., $d(f(p), x_0) < M$, $\forall p \in E$.

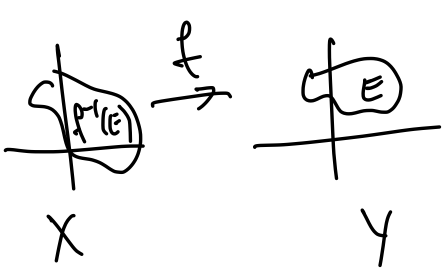


Thm. If $f: X \to Y$ is continuous, and X is compact, then $f(X)$ is compact.

Pf: Let $\{V_\alpha\}$ be an open cover of $f(X)$. By a theorem from last lecture (Lecture 14, Video 6), the preimages $f^{-1}(V_\alpha) \subset X$ are open. Now, since X is compact and $X \subset \bigcup_\alpha f^{-1}(V_\alpha)$, there exists a <u>finite subcover</u>, i.e., there exist $\alpha_1, \ldots, \alpha_n$ s.t.

$$X \subset f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n})$$

Since $f(f^{-1}(E)) \subset E$, by the above, we have



$$f(X) \subset f\left(f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n})\right)$$

$$\subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

This is a finite subcover of $\{V_\alpha\}_i$ proving that $f(X)$ is compact. $\square$

Recall the Heine-Borel Theorem (Lecture 6, Video 1):

$$E \subset \mathbb{R}^k \text{ is } \underline{\text{compact}} \iff E \subset \mathbb{R}^k \text{ is } \underline{\text{closed and bounded}}$$

<u>Cor</u>: If $f: X \to \mathbb{R}^k$ continuous and X is compact, then $f(X) \subset \mathbb{R}^k$ is closed and bounded.

In particular, if $k=1$, we have:

If $f: X \to \mathbb{R}$ is continuous and X is compact, then $\exists\, p, q \in X$ such that

$$f(p) = \sup_{x \in X} f(x) \qquad f(q) = \inf_{x \in X} f(x).$$

<u>Thm.</u> If $f: X \to Y$ is continuous and injective, and X is compact, then the inverse map $f^{-1}: f(X) \to X$ defined by

$$f^{-1}(f(x)) = x, \qquad x \in X$$

is a continuous map.

Pf: By a result of last lecture (Lecture 14, Video 6), it suffices to show that $f(V)$ is open in Y whenever $V \subset X$ is open. Given $V \subset X$ open, $V^c = X \setminus V$ is closed in the compact metric space X, hence V^c is compact. Now, by Theorem above (Video 1 of today!), $f(V^c)$ is compact. Since $f(V) = f(X) \setminus f(V^c)$ because f is injective, we have that $f(V)$ is the complement of a closed set, i.e., $f(V)$ is open, as desired. $\square$

Uniform continuity

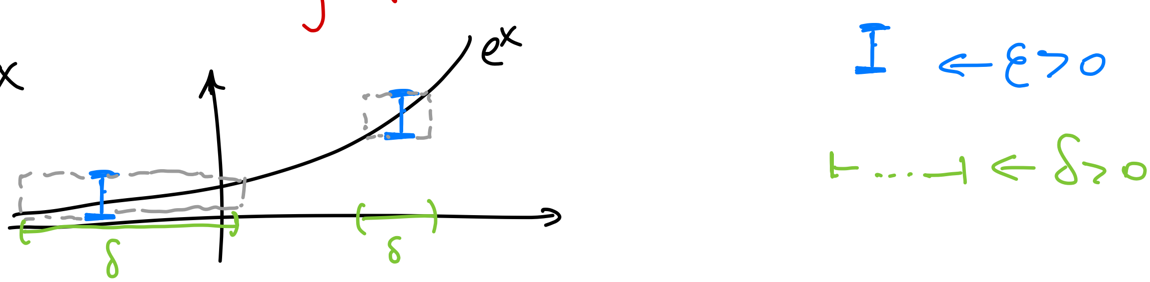
Def: $f: X \to Y$ is <u>uniformly continuous</u> if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$$d_X(p,q) < \delta \implies d_Y(f(p), f(q)) < \varepsilon.$$

Remark: The above is (strictly) <u>stronger</u> than saying $f: X \to Y$ is continuous: the choice of δ must work <u>everywhere</u> without reference to any point of the domain.

$f: \mathbb{R} \to \mathbb{R}$

Ex: $f(x) = e^x$ is <u>not</u> unif. cont.



Theorem: If $f: X \to Y$ is continuous and X is compact, then $f: X \to Y$ is uniformly continuous.

Pf: Given $\varepsilon > 0$, since f is continuous, for each $p \in X$,

$$\exists \underbrace{\phi(p)}_{\substack{\text{"the" } \delta \text{ that works} \\ \text{for continuity} \\ \text{at that } p \in X}} > 0 \text{ s.t.}$$

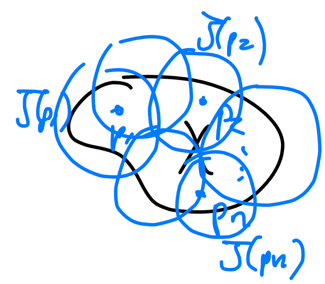
$$q \in X, \; d_X(p,q) < \phi(p) \overset{(*)}{\Longrightarrow} d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$$

Let $J(p) = \left\{ q \in X : d_X(p,q) < \frac{\phi(p)}{2} \right\}$; note $p \in J(p)$, and $J(p)$ is open in X, so

$\bigcup_{p \in X} J(p) = X$, so $\{J(p)\}_{p \in X}$ is an open cover of X.

Since X is compact, there exists a finite subcover:

$$X \subset J(p_1) \cup \dots \cup J(p_n)$$



Set $\delta := \frac{1}{2} \min \{ \phi(p_1), \dots, \phi(p_n) \} > 0$.

b/c only finitely many $\{p_i\}$.

Let's verify that the above δ works to prove uniform continuity of $f: X \to Y$.

Given $p, q \in X$, $d_X(p,q) < \delta$, we know $\exists m \in \mathbb{N}$, $1 \le m \le n$ s.t. $p \in J(p_m)$, i.e., $d_X(p, p_m) < \frac{1}{2} \phi(p_m)$.

We also have

$$d_X(q, p_m) \le \underbrace{d_X(p,q)}_{<\delta} + \underbrace{d_X(p,p_m)}_{<\frac{1}{2}\phi(p_m)} < \delta + \frac{\phi(p_m)}{2} \le \phi(p_m)$$

So, by (*) we have

$$d_Y(f(p), f(q)) \le d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) \overset{(*)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

This establishes unif. continuity of $f : X \to Y$. $\square$

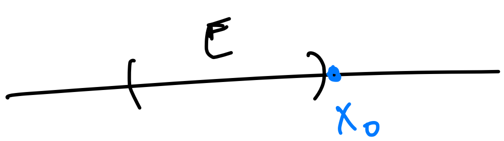
Compactness is necessary on all of the above results:

Suppose $E \subset \mathbb{R}$ is not compact : E is unbounded or E is not closed

Heine-Borel

1. Example of $f : E \subset \mathbb{R} \to \mathbb{R}$ continuous which is not bounded.

- If E is bounded and not closed : then $\exists x_0 \in \mathbb{R}$ a limit point of E s.t. $x_0 \notin E$



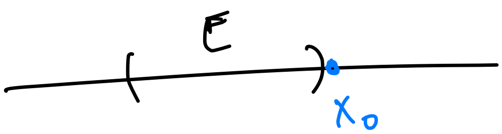
Take $f : E \to \mathbb{R}$, $f(x) = \dfrac{1}{x - x_0}$

f is continuous and unbounded : $|f(x)| \nearrow +\infty$ if $x \to x_0$.

- If E is unbounded, just take $f(x) = x$.

2. Example of $f: E \subset \mathbb{R} \to \mathbb{R}$ which is continuous and bounded but f has no maximum.

- If E is bounded and not closed: then $\exists x_0 \in \mathbb{R}$ a limit point of E s.t. $x_0 \notin E$. Take $f: E \subset \mathbb{R} \to \mathbb{R}$



$$f(x) = \frac{1}{1 + (x - x_0)^2} \quad \text{(is continuous)}$$

$\forall x \in \mathbb{R}$, $0 \leq f(x) \leq 1$ so f is bounded; clearly

$\sup_{x \in E} f(x) = 1$ but $f(x) < 1$ because $x_0 \notin E$.

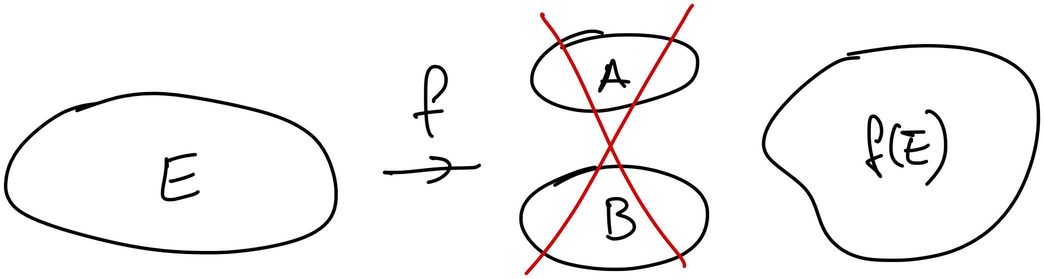
- If E is unbounded, then take $f(x) = \dfrac{x^2}{1+x^2}$.

Then f is continuous, $0 \leq f(x) \leq 1$, $\forall x \in \mathbb{R}$,

$\sup_{x \in E} f(x) = 1$, but $f(x) < 1$, $\forall x \in E$.

b/c we can let $|x| \to \infty$, $f(x) \to 1$

Continuity and Connectedness

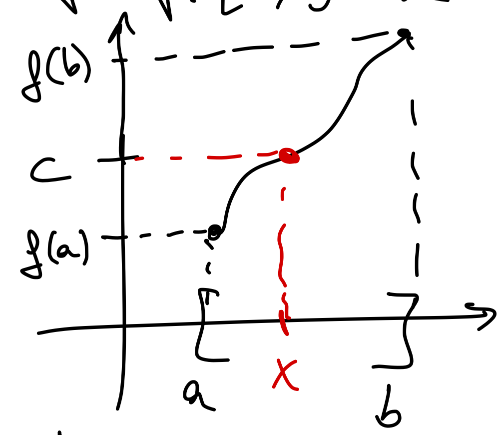


Theorem: If $f: X \to Y$ is continuous and $E \subset X$ is connected, then $f(E)$ is connected.

Pf: By contradiction, suppose $f(E) = A \cup B$ where $A, B \subset Y$ are separated. Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. Then $E = G \cup H$ and $G \neq \phi$, $H \neq \phi$.

Since $A \subset \overline{A}$, we have $G \subset f^{-1}(A) \subset f^{-1}(\overline{A})$.
Since f is continuous, $f^{-1}(\overline{A})$ is closed. Therefore, $\overline{G} \subset f^{-1}(\overline{A})$, i.e., $f(\overline{G}) \subset \overline{A}$. We have $f(H) = B$ and $\overline{A} \cap B = \phi$ (b/c A, B are separated), so $\overline{G} \cap H = \phi$. Reversing roles of G, H and A, B, one arrives at the conclusion that $G \cap \overline{H} = \phi$. But $E = G \cup H$ is connected, so this cannot hapen. $\square$

Corollary: ("Intermediate Value Theorem"). If $f:[a,b]\to\mathbb{R}$ is continuous, $f(a)\le f(b)$, then for all $c\in\mathbb{R}$, $f(a)\le c\le f(b)$, there exists $x\in[a,b]$ s.t. $f(x)=c$.



Pf: By a theorem we have seen before (Lecture 6, Video 5), the connected subsets of $\mathbb{R}$ are intervals. By the Theorem above, $f(\underbrace{[a,b]}_{\text{connected}})\subset\mathbb{R}$ is connected, therefore, it is an interval, so any $c\in\mathbb{R}$ with $f(a)\le c\le f(b)$ is also a member of $f([a,b])$, that is, of the form $c=f(x)$ for some $x\in[a,b]$. $\square$

Remark: Can use this to solve complicated equations without doing any algebra!
(Check HW5 for an example...)