Recall: $f_n : E \to \mathbb{R}$ converges uniformly to $f_\infty : E \to \mathbb{R}$ if $\forall \varepsilon > 0$ $\exists N = N(\varepsilon) \in \mathbb{N}$ s.t. $n \geq N$

$$|f_n(x) - f_\infty(x)| \leq \varepsilon, \qquad \forall x \in E$$

Thm: Suppose $f_n \to f_\infty$ uniformly, let x be a limit point of E and $\lim_{t \to x} f_n(t) = A_n$, $\forall n \in \mathbb{N}$. Then $\{A_n\}$ converges and $\lim_{t \to x} f_\infty(t) = \lim_{n \to \infty} A_n$.

(In other words: We may exchange the order of the limits:

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t) \quad)$$

Pf: Given $\varepsilon > 0$, since $f_n \to f_\infty$ unif., $\exists N \in \mathbb{N}$ s.t. if $n, m \geq N$, $t \in E$, then $|f_n(t) - f_m(t)| \leq \varepsilon$. Letting $t \to x$ we find $|A_n - A_m| \leq \varepsilon$. Thus $\{A_n\}$ is a Cauchy seq., and hence $A_n \to A_\infty$. Now,

$$|f_\infty(t) - A_\infty| \leq \underbrace{|f_\infty(t) - f_n(t)|}_{\leq \varepsilon/3} + \underbrace{|f_n(t) - A_n|}_{\leq \varepsilon/3} + \underbrace{|A_n - A_\infty|}_{\leq \varepsilon/3}$$

Choose $n \in \mathbb{N}$ large enough so that $|f_\infty(t) - f_n(t)| \leq \varepsilon/3$; and so that $|A_n - A_\infty| \leq \varepsilon/3$. For this n, we choose a neighborhood $U \ni x$ s.t. $|f_n(t) - A_n| \leq \varepsilon/3$, $\forall t \in U \cap E$, $t \neq x$.

Altogether, we have $|f_\infty(t) - A_\infty| \le \varepsilon$, $\forall t \in U \cap E$, $t \neq x$. Since $\varepsilon > 0$ was arbitrary, this gives $\lim_{t \to x} f_\infty(t) = A_\infty$.

$\square$

Corollary: If $f_n : E \to \mathbb{R}$ is continuous $\forall n \in \mathbb{N}$ and $f_n \to f_\infty$ uniformly, then $f_\infty : E \to \mathbb{R}$ is continuous.

Pf: From Thm above, we know that if x is a limit pt of E:

$$\lim_{t \to x} f_\infty(t) = \lim_{t \to x} \underbrace{\lim_{n \to \infty} f_n(t)}_{f_\infty(t)} = \lim_{n \to \infty} \underbrace{\lim_{t \to x} f_n(t)}_{f_n(x)} = f_\infty(x).$$

cont. of f_n

$\square$

Remark:

- Uniform convergence is necessary (counter-examples given in Lecture 23).
- If f_n are cont. and $f_n \to f_\infty$ and f_∞ is cont., in general, the convergence need not be uniform. However:

Thm. Suppose K is compact, and

(i) $f_n : K \to \mathbb{R}$ are continuous $\forall n \in \mathbb{N}$

(ii) $f_n \to f_\infty$ converges pointwise to $f_\infty : K \to \mathbb{R}$ and f_∞ is continuous

(iii) $f_n(x) \geq f_{n+1}(x)$, $\forall x \in K$, $\forall n \in \mathbb{N}$

Then $f_n \to f_\infty$ uniformly.

Pf: Set $g_n = f_n - f_\infty$. Then g_n are continuous, $g_n \to 0$ pointwise, and $g_n \geq g_{n+1}$. We want to show $g_n \to 0$ uniformly.

Given $\varepsilon > 0$, let $K_n = \{x \in K : g_n(x) \geq \varepsilon\}$. Since g_n is cont., K_n is closed and hence compact. Moreover, $g_n \geq g_{n+1}$ implies $K_n \supset K_{n+1}$. Fix $x \in K$. Since $g_n(x) \to 0$, we see that $x \notin K_n$ if n is sufficiently large. Thus $x \notin \bigcap_{n\in\mathbb{N}} K_n$. Since $x \in K$ was arbitrary, it follows that $\bigcap_{n\in\mathbb{N}} K_n = \emptyset$. Thus, $K_N = \emptyset$ for some $N \in \mathbb{N}$ (by Video 5 of Lecture 5). In other words, $0 \leq g_n(x) < \varepsilon$ $\forall x \in K$, if $n \geq N$. Therefore $g_n \to 0$ uniformly. $\square$

<u>The vector space $\mathcal{C}(X,\mathbb{R})$</u>

$\mathcal{C}(X,\mathbb{R}) = \{ f : X \to \mathbb{R}$ continuous and bounded functions$\}$

$f, g \in \mathcal{C}(X,\mathbb{R})$ $\qquad$ $f+g \in \mathcal{C}(X,\mathbb{R})$ ← $(f+g)(x) = f(x)+g(x)$

$a \in \mathbb{R}$ $\qquad$ $a\cdot f \in \mathcal{C}(X,\mathbb{R})$ ← $(a\cdot f)(x) = a\cdot f(x)$.

So $\mathcal{C}(X,\mathbb{R})$ is a <u>real vector space</u> (of infinite dimension)

(<u>Rmk</u>: If X is compact, then "bounded" follows from "continuous".)

Def: $\|f\| = \sup_{x\in X} |f(x)|$ is the "sup-norm" ($\|f\| < \infty$ b/c f is bounded)

Prop: $(\mathcal{C}(X,\mathbb{R}), \|\cdot\|)$ is a normed vector space.

Pf: (1) $\|f\| \geq 0$ $\forall f \in \mathcal{C}(X,\mathbb{R})$ is obviously true.

$$\|f\| = 0 \iff |f(x)| = 0, \ \forall x \in X$$
$$\iff f(x) = 0, \ \forall x \in X$$
$$\iff f = 0$$

(2) Setting $h = f+g$, we have:

$$|h(x)| \leq |f(x)| + |g(x)| \leq \sup_{x\in X} f(x)+g(x) \overset{\text{HW1}}{\leq} \underbrace{\sup_{x\in X} f(x)}_{\|f\|} + \underbrace{\sup_{x\in X} g(x)}_{\|g\|}$$

So $\underbrace{\sup_{x\in X} |h(x)|}_{\|h\| = \|f+g\|} \leq \|f\| + \|g\|$

This proves the triangle inequality

$$\|f+g\| \leq \|f\| + \|g\|. \qquad \square$$

Note: This norm makes $\mathcal{C}(X,\mathbb{R})$ into a metric space itself, with $d(f,g) = \|f-g\|$.

Note: Given $f_n \in \mathscr{C}(X,\mathbb{R})$ and $f_\infty \in \mathscr{C}(X,\mathbb{R})$, $f_n \to f_\infty$ converges (in metric space sense) if and only if $d(f_n, f_\infty) = \|f_n - f_\infty\| \to 0$, which, in turn, is equivalent to $f_n \to f_\infty$ uniformly.

Thm: $\mathscr{C}(X,\mathbb{R})$ is a complete metric space.

Pf: Let $\{f_n\}$ be a Cauchy seq., i.e., $\forall \varepsilon > 0\ \exists N$ s.t. $\|f_n - f_m\| < \varepsilon$, if $n, m \geq N$. By the Cauchy criterion for unif. conv. (Video 5 of Lecture 23), there is a limit function $f_\infty : X \to \mathbb{R}$ and $f_n \to f_\infty$ uniformly. By Corollary above, f_∞ is continuous. Moreover, f_∞ is bounded because $\exists n$ s.t. $|f_\infty(x) - f_n(x)| < 1$ for all $x \in X$. So $f_\infty \in \mathscr{C}(X,\mathbb{R})$. Since $f_n \to f_\infty$ uniformly, $\|f_n - f_\infty\| \to 0$. $\square$

Uniform convergence and integration

Thm. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ for all $n \in \mathbb{N}$ and $f_n \to f_\infty$ uniformly on $[a,b]$. Then $f_\infty \in \mathscr{R}(\alpha)$ and

$$\int_a^b f_\infty \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha$$

Pf: Define $\varepsilon_n = \|f_n - f_\infty\| = \sup\limits_{x\in[a,b]} |f_n(x) - f_\infty(x)| \underset{n\to\infty}{\searrow} 0.$

Then, for all $n \in \mathbb{N}$,

$$f_n - \varepsilon_n \le f_\infty \le f_n + \varepsilon_n \qquad \forall x \in [a,b].$$

So

$$\int_a^b (f_n - \varepsilon_n)\,d\alpha \le \underline{\int_a^b} f_\infty\,d\alpha \le \overline{\int_a^b} f_\infty\,d\alpha \le \int_a^b (f_n + \varepsilon_n)\,d\alpha$$

So:

$$0 \le \overline{\int_a^b} f_\infty\,d\alpha - \underline{\int_a^b} f_\infty\,d\alpha \le \int_a^b (f_n + \varepsilon_n)\,d\alpha - \int_a^b (f_n - \varepsilon_n)\,d\alpha$$

$$= \int_a^b \cancel{(f_n - f_n)} + 2\varepsilon_n\,d\alpha$$

$$= \int_a^b 2\varepsilon_n\,d\alpha = 2\varepsilon_n(\alpha(b) - \alpha(a))$$

Since $\lim\limits_{n\to\infty} \varepsilon_n = 0$, we have $\overline{\int_a^b} f_\infty\,d\alpha = \underline{\int_a^b} f_\infty\,d\alpha$, i.e. $f_\infty \in \mathcal{R}(\alpha)$. Moreover,

$$\int_a^b f_\infty\,d\alpha \le \int_a^b (f_n + \varepsilon_n)\,d\alpha \implies \int_a^b (f_\infty - f_n)\,d\alpha \le \underbrace{\int_a^b \varepsilon_n\,d\alpha}_{\varepsilon_n(\alpha(b)-\alpha(a))}$$

So

$$\left| \int_a^b f_\infty\,d\alpha - \int_a^b f_n\,d\alpha \right| = \left| \int_a^b (f_\infty - f_n)\,d\alpha \right| \le \varepsilon_n(\alpha(b) - \alpha(a))$$

Therefore, $\displaystyle\int_a^b f_\infty \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha.$ $\square$

<u>Corollary</u>: If $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ and consider the series

$$f(x) = \sum_{n=1}^{+\infty} f_n(x), \qquad x \in [a,b].$$

If the above series converges <u>uniformly</u> on $[a,b]$, then one may perform integration term-by-term:

$$\int_a^b f \, d\alpha = \sum_{n=1}^{+\infty} \int_a^b f_n \, d\alpha$$

<u>Pf</u>: $\displaystyle F_N(x) := \sum_{n=1}^{N} f_n(x)$, $\quad F_N \longrightarrow f$ uniformly.

By Thm: $f \in \mathcal{R}(\alpha)$ and

$$\int_a^b f \, d\alpha = \lim_{N\to\infty} \int_a^b F_N \, d\alpha = \lim_{N\to\infty} \int_a^b \sum_{n=1}^{N} f_n(x) \, d\alpha$$

Finite sum inside lim

$$\overset{\downarrow}{=} \lim_{N\to\infty} \sum_{n=1}^{N} \int_a^b f_n(x) \, d\alpha = \sum_{n=1}^{+\infty} \int_a^b f_n \, d\alpha.$$

$\square$

Uniform Convergence and differentiation

<u>Thm.</u> Suppose $f_n : [a,b] \to \mathbb{R}$ are differentiable on $[a,b]$, and $\exists x_0 \in [a,b]$ s.t. $\{f_n(x_0)\}$ is a convergent sequence. If $f_n' \to g$ uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly to $f_\infty : [a,b] \to \mathbb{R}$ and

$$f_\infty'(x) = \lim_{n\to\infty} f_n'(x), \quad \forall x \in [a,b].$$

<u>Pf:</u> Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be s.t. $n, m \geq N$

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \quad (**)$$

and $$|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)}, \quad \forall t \in [a,b]$$

By the Mean Value Thm, applied to $\overbrace{f_n - f_m}^{\psi_{n,m}}$,

$$\Big|\underbrace{f_n(x) - f_m(x)}_{\psi_{n,m}(x)} - \big(\underbrace{f_n(t) - f_m(t)}_{\psi_{n,m}(t)}\big)\Big| = \big|\underbrace{f_n'(t_0) - f_m'(t_0)}_{\psi_{n,m}'(t_0)}\big|\,|x-t|$$

$\exists\, t_0$ between t and x in $[a,b]$.

$$\overset{(*)}{\leq} \frac{\varepsilon|x-t|}{2(b-a)} \overset{(*)}{\leq} \frac{\varepsilon}{2}.$$

for all $t, x \in [a,b]$; if $m, n \geq N$. Moreover, by the triangle ineq.:

$$\underbrace{|f_n(x) - f_m(x)|}_{\psi_{n,m}(x)} \leq \underbrace{\Big|\underbrace{f_n(x) - f_m(x)}_{\psi_{n,m}(x)} - \big(\overbrace{f_n(x_0) - f_m(x_0)}^{\psi_{n,m}(x_0)}\big)\Big|}_{\leq \varepsilon/2 \quad (*)} + \underbrace{\big|\underbrace{f_n(x_0) - f_m(x_0)}_{\psi_{n,m}(x_0)}\big|}_{\leq \varepsilon/2 \quad (**)}$$

$$< \varepsilon.$$

Therefore, f_n converges uniformly (by Cauchy Criterion).
Let $f_\infty : [a,b] \to \mathbb{R}$ be its limit:

$$f_\infty(x) = \lim_{n \to \infty} f_n(x), \qquad \forall x \in [a,b]$$

Given $x \in [a,b]$, $n \in \mathbb{N}$, let (for $t \neq x$):

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x} \quad \text{and} \quad \phi_\infty(t) = \frac{f_\infty(t) - f_\infty(x)}{t - x}.$$

By definition, $\lim_{t \to x} \phi_n(t) = f_n'(x)$. By (*),

$$|\phi_n(t) - \phi_m(t)| \leq \frac{\varepsilon}{2(b-a)} \quad \text{if } n, m \geq N.$$

So ϕ_n converges uniformly (by Cauchy criterion) for $t \neq x$.

Since $f_n \to f_\infty$, we conclude that:

$$\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \frac{f_n(t)-f_n(x)}{t-x} = \frac{\lim_{n\to\infty} f_n(t) - \lim_{n\to\infty} f_n(x)}{t-x}$$

$$= \frac{f_\infty(t)-f_\infty(x)}{t-x} = \phi_\infty(t)$$

Applying first Thm of today's lecture to $\phi_n \xrightarrow{\text{unif.}} \phi_\infty$,

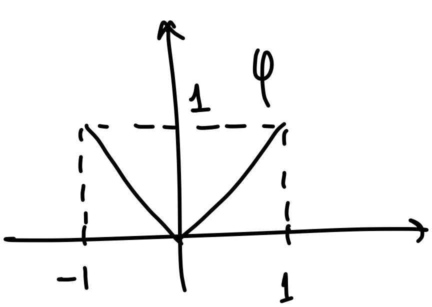
$$\lim_{t\to x} \underbrace{\lim_{n\to\infty} \phi_n(t)}_{\phi_\infty(t)} = \lim_{n\to\infty} \underbrace{\lim_{t\to x} \phi_n(t)}_{f_n'(x)}$$

So

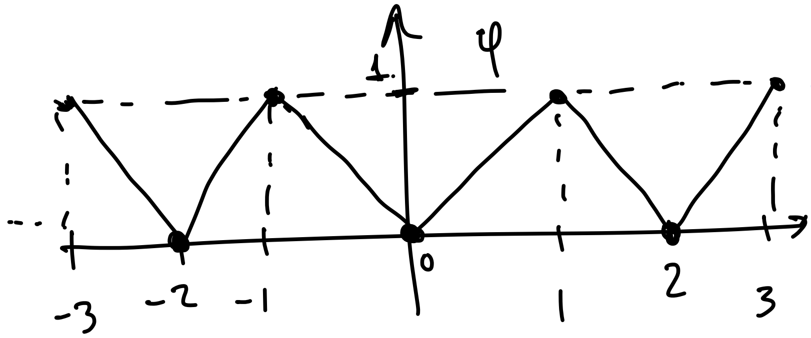
$$f_\infty'(x) = \lim_{t\to x} \phi_\infty(t) = \lim_{n\to\infty} f_n'(x). \qquad \square$$

<u>Thm</u>: There exists a function $f:\mathbb{R}\to\mathbb{R}$ which is everywhere continuous, but nowhere differentiable.

<u>Pf</u>: Define $\varphi(x)=|x|$ if $x\in[-1,1]$. Extend φ to a function $\varphi:\mathbb{R}\to\mathbb{R}$ by making it periodic, with period 2:



$$\varphi(x) = \varphi(x+2), \quad \forall x\in\mathbb{R}.$$



$\varphi(x) = 1 \iff x \in \mathbb{Z}$, odd

$\varphi(x) = 0 \iff x \in \mathbb{Z}$, even

Moreover, φ is "Lipschitz with Lipschitz constant 1":

$$(*) \quad |\varphi(s) - \varphi(t)| \le |s - t|, \quad \forall s,t \in \mathbb{R}.$$

In particular, φ is continuous. Define $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \sum_{n=0}^{+\infty} \underbrace{\left(\frac{3}{4}\right)^n \varphi(4^n x)}_{f_n(x)}.$$

Since $0 \le \varphi(t) \le 1$, we have $|f_n(x)| \le \left(\frac{3}{4}\right)^n$.

Let $M_n = \left(\frac{3}{4}\right)^n$, by Video 6 of Lecture 23, we have:

$$\sum_{n=1}^{+\infty} M_n = \sum_{n=1}^{+\infty} \left(\frac{3}{4}\right)^n = \frac{3/4}{1 - 3/4} = \frac{3/4}{1/4} = 3 < +\infty$$

$$\Downarrow$$

$f(x) = \sum_{n=1}^{+\infty} f_n(x)$ converges uniformly.

By first theorem of today, since each f_n is continuous, we have that $f(x) = \sum_{n=1}^{+\infty} f_n(x)$ is continuous.

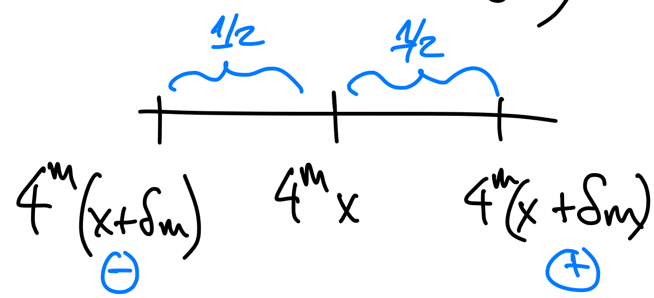
Let us now show $f(x)$ is nowhere differentiable.

Fix $x \in \mathbb{R}$ and $m \in \mathbb{N}$. Choose $\delta_m = \pm \frac{1}{2} \cdot \frac{1}{4^m}$, where $\pm$ is chosen in such a way that no integer lies between $4^m x$ and $4^m(x+\delta_m)$.

(This is possible b/c $\left|4^m x - 4^m(x+\delta_m)\right| = \left|4^m \delta_m\right| = \frac{1}{2}$.)

Define

$$\gamma_n = \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}$$



If $n>m$, then $\underset{\substack{\| \\ 2^p \\ p=2(n-m)\geq 2}}{4^n \delta_m = 4^{\overset{>0}{n-m}}}\underbrace{(4^m \delta_m)}_{= \pm 1/2}$ is an even integer

so $\varphi(4^n x + \underbrace{4^n \delta_m}_{\text{even integer}}) = \varphi(4^n x)$ because φ is periodic with period 2.

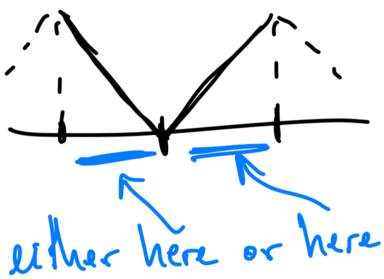
Thus $\gamma_n = 0$.

If $0 \leq n \leq m$, by $\circledast$,

$$|\gamma_n| = \frac{\left|\varphi(4^n(x+\delta_m)) - \varphi(4^n x)\right|}{|\delta_m|} \overset{\circledast}{\leq} \frac{\left|\cancel{4^n x} + 4^n \delta_m - \cancel{4^n x}\right|}{|\delta_m|}$$

$$\leq \frac{4^n |\delta_m|}{|\delta_m|} = 4^n.$$

In particular, if $n = m$, then $\left|\varphi\left(4^n(x+\delta_m)\right) - \varphi(4^n x)\right| = \frac{1}{2}$; because there are no integers between $4^m(x+\delta_m)$ and $4^m x$ so, since $4^m \delta_m = \pm \frac{1}{2}$,



$$\left|\varphi\left(4^m(x+\delta_m)\right) - \varphi(4^m x)\right| = |\delta_m| = \frac{1}{2}.$$

Altogether

$$|\gamma_n| = \begin{cases} \leq 4^n & 0 \leq n < m \\ 4^n & n = m \\ 0 & n > m \end{cases}$$

Thus

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \left|\frac{\sum_{n=0}^{+\infty}\left(\frac{3}{4}\right)^n \varphi\left(4^n(x+\delta_m)\right) - \sum_{n=0}^{+\infty}\left(\frac{3}{4}\right)^n \varphi(4^n x)}{\delta_m}\right|$$

$$= \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \underbrace{\frac{\varphi\left(4^n(x+\delta_m)\right) - \varphi(4^n x)}{\delta_m}}_{\gamma_n}\right|$$

$$\geq \underbrace{\left(\frac{3}{4}\right)^m \cdot 4^m}_{n=m} - \underbrace{\sum_{n=0}^{m-1} 3^n}_{n \leq m-1}$$

$$= \frac{1}{2}\left(3^m + 1\right) \nearrow +\infty \quad \text{as } m \to \infty$$

$$\left(\text{cf. } \delta_m \searrow 0 \quad \text{as } m \to +\infty\right)$$

So the limit of $\dfrac{f(x+\delta) - f(x)}{\delta}$ as $\delta \to 0$ diverges, and hence $f(x)$ is not differentiable anywhere.

$\square$