MAT 320 Lecture 3 9/2/2020

Basic Notions:

Def: A, B are sets, $f: A \to B$ function is

(1) injective (or "1-1") if $\forall x, y \in A$, $x \neq y \Rightarrow f(x) \neq f(y)$

(2) surjective (or "onto") if $\forall b \in B$, $\exists x \in A$ s.t. $b = f(x)$

(3) bijective if it is injective and surjective. In this case, f has an inverse f^{-1}.

Def: A and B have the same cardinality if $\exists f: A \to B$ a bijective map ("bijection").

Notation: $\# A = \# B$, $|A| = |B|$

intuitively, this means that A and B have the same number of elements.

Finite / Countable / Uncountable.

$J_1 = \{1\}$
$J_2 = \{1, 2\}$
$J_3 = \{1, 2, 3\}$
$\vdots$

Notation: For all $n \in \mathbb{N}$, let $J_n := \{1, \dots, n\}$.

$$\mathbb{N} = \{1, 2, 3, \dots\} = \bigcup_{n \in \mathbb{N}} J_n$$

Def: A set A is

(1) finite if $\exists n \in \mathbb{N}$ and $f : A \to J_n$ a bijection.
$(\# A = n)$

(2) infinite if A is not finite.

(3) countable if $\exists f : A \to \mathbb{N}$ bijection,

(4) uncountable if it is not finite nor countable

Examples: $\mathbb{N}$ is countable and infinite.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is also countable and infinite (note $\mathbb{N} \subset \mathbb{Z}$).

$\left(\begin{array}{c} A \subset B \\ \#A \leq \#B \end{array} \right)$

exercise

$\exists f : \mathbb{Z} \to \mathbb{N}$ bijection

$$g : \mathbb{N} \to \mathbb{Z}, \quad g(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even} \\[2ex] -\dfrac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

start here

$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

$$\begin{array}{cccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots \\ g\downarrow\uparrow f & \downarrow\uparrow & \downarrow\uparrow & \downarrow\uparrow & \downarrow\uparrow & \downarrow\uparrow & \downarrow\uparrow & \\ 0, & 1, & -1, & 2, & -2, & 3, & -3, & \end{array}$$

$$g(1) = -\frac{1-1}{2} = \underline{\underline{0}}$$

$$g(2) = \frac{2}{2} = \underline{\underline{1}}$$

$$g(3) = -\frac{3-1}{2} = \underline{\underline{-1}}$$

$$g(4) = \frac{4}{2} = \underline{\underline{2}} \quad \ldots$$

Note that g is a bijection. Let f be the inverse of g, which is also a bijection.

$(f(g(n)) = n, \quad g(f(x)) = x)$

Alternative definition of infinite set:

Even though $\mathbb{N} \subsetneq \mathbb{Z}$, they have the same cardinality: $\exists\, f : \mathbb{Z} \to \mathbb{N}$ a bijection.

Thm: A set A is infinite if and only if there exists a proper subset $E \subsetneq A$ and a bijection $f : A \to E$.

Pf: Suppose $\exists\, E \subsetneq A$ and $f : A \to E$ a bijection. If $\exists\, n \in \mathbb{N}$ and $\varphi : A \to J_n$ a bijection, let $n_0 \in \mathbb{N}$ be the smallest such n and $\varphi : A \to J_{n_0}$.

Consider $\varphi|_E : E \to J_{n_0}$. Then $\exists\, m < n_0$ such that $\operatorname{Im} \varphi|_E = \varphi(E) \cong J_m$. Then $(\varphi \circ f) : A \to J_m$

$$
\begin{tikzcd}
A \arrow[r, "\varphi", "\cong"'] \arrow[d, "f"', "\cong"] \arrow[dr, dashed, bend right=60, "(\varphi\circ f)"'] & J_{n_0} \\
E \arrow[r, "\varphi|_E", "\cong"'] & J_m \arrow[u, hook]
\end{tikzcd}
$$

would be a bijection from A to J_m, $m < n_0$, contradicting minimality of n_0. Thus, A is <u>infinite</u>.

Converse is an exercise. $\square$

<u>Hint</u>: Take a countable subset $E' \subset A$ and let $E' = \{x_1, x_2, \dots\}$. Then, consider the map $f : E' \to E'$ given by $f(x_i) = x_{i+1}$, $\forall i \in \mathbb{N}$.

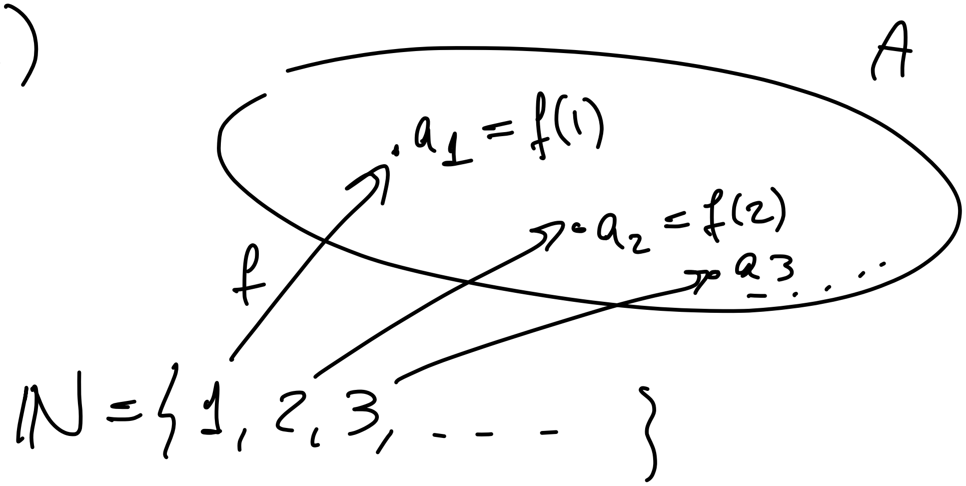
Conclude that $E' \cong \{x_2, x_3, \dots\}$
$= E' \setminus \{x_1\}$.

(Can extend $f : E' \to E' \setminus \{x_1\}$ to $f : A \to \underbrace{A \setminus \{x_1\}}_{=:E}$ by the identity.)

Sequences:

Def: A sequence in the set A is a function $f: \mathbb{N} \to A$

Notation: $a_n = f(n)$



Thm: Every infinite subset of a countable set A is also countable.

Pf: Since A is countable, let $f: A \to \mathbb{N}$ be a bijection.

Write $x_n \in A$ for the element of A mapped to $n \in \mathbb{N}$ by the bijection f.

$A = \{x_1, x_2, x_3, \ldots\}$

Let $E \subset A$ be an <u>infinite</u> subset. Define a sequence $\{n_k\} \subset \mathbb{N}$ as follows:

Let n_1 be the smallest number s.t. $x_{n_1} \in E$

$\vdots$

after choosing $n_1, n_2, \ldots, n_{k-1}$, let n_k be the smallest number $\geq n_{k-1}$ such that $x_{n_k} \in E$. Since E is infinite, this defines an infinite sequence $\{n_k\}$; note $E = \{x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots\}$

Let $g : E \to \mathbb{N}$ be s.t. $g(x_{n_k}) = k$. Observe that g is a bijection. Therefore E is countable. $\square$

More properties of countable sets:

Thm: (1) The countable union of countable sets is countable

(2) The (finite) cartesian product of countable sets is countable.

Pf: (1) Let $E_1, E_2, \ldots, E_n, \ldots$ be countable sets for each $n \in \mathbb{N}$. We need to show that

$$E := \bigcup_{n\in\mathbb{N}} E_n = E_1 \cup E_2 \cup E_3 \cup \ldots = \bigcup_{n=1}^{\infty} E_n$$

is countable.

Since each E_n is countable, $\exists f_n : E_n \to \mathbb{N}$ bij.

$$E_1 = \{ x_{11}, \; x_{12}, \; x_{13}, \; \ldots \}$$

$$E_2 = \{ x_{21}, \; x_{22}, \; x_{23}, \; \ldots \}$$

$$E_3 = \{ x_{31}, \; x_{32}, \; x_{33}, \; \ldots \}$$

$$\left(\begin{matrix} x_{ij} \in E_i \\ f_i(x_{ij}) = j \end{matrix} \right)$$

We can enumerate the elements of E as follows:

$$\begin{array}{lcccccccccccc} E: & x_{11}, & x_{21}, & x_{12}, & x_{31}, & x_{22}, & x_{13}, & x_{41}, & x_{32}, & x_{23}, & x_{14}, & \ldots \\ \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \end{array}$$

This shows that E is countable.

(2) Now, we would like to show that

$\underbrace{E_1 \times E_2 \times E_3 \times \ldots \times E_k}_{}$ are countable for all $k \in \mathbb{N}$

$$= \prod_{i=1}^{k} E_i = \{ (x_1, x_2, \ldots, x_k) : \; x_i \in E_i, \; i = 1, \ldots, k \}.$$

Using the bijections $f_n : E_n \to \mathbb{N}$, it suffices to show that

$$\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{k\text{-fold cartesian product of naturals}} \text{ is countable.}$$

$$= \{(n_1, n_2, \dots, n_k) : n_i \in \mathbb{N},\ i = 1, \dots, k\}$$

For $k = 1$ this is clear since $\mathbb{N}^1 = \mathbb{N}$.

Suppose (by induction) that $\mathbb{N}^{k-1}$ is countable.

Since $\mathbb{N}^k = \mathbb{N}^{k-1} \times \mathbb{N}$, the elements of $\mathbb{N}^k$ are

$$(\underbrace{n_1, \dots, n_{k-1}}_{(n_1, \dots, n_{k-1}) \in \mathbb{N}^{k-1}}, \underbrace{n_k}_{\in \mathbb{N}}) \in \mathbb{N}^k$$

countable, by induction.

Thus: $\mathbb{N}^k = \bigcup_{n \in \mathbb{N}} \mathbb{N}^{k-1}$ is a countable union of countable sets. This is countable by (1). $\square$

Application:

Cor: $\mathbb{Q}$ is countable.

Pf: There is a "copy" of $\mathbb{Q}$ inside $\mathbb{Z} \times \mathbb{N}$

$$\mathbb{Q} \ni \frac{p}{q} \longleftrightarrow p \in \mathbb{Z},\ q \in \mathbb{N}$$

Since $\mathbb{Z}$, $\mathbb{N}$ are countable, so is $\mathbb{Z} \times \mathbb{N}$, and hence $\mathbb{Q}$ is also countable. $\square$

(There is a surjection $\mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Q}$)

Example: $\{0,1\}^{\mathbb{N}} := \{ f : \mathbb{N} \longrightarrow \{0,1\} \}$ is uncountable.

(the set of sequences of 0's and 1's.)

Pf: ("Cantor diagonal"):

Let $E \subset \{0,1\}^{\mathbb{N}}$ be a countable subset, say

$$E = \{ s_1, s_2, s_3, \dots, s_n, \dots \}$$

1st element of S_1

$S_1 : \ \boxed{0}\ \ 0\ \ 1\ \ 0\ \ 1\ \ 1\ \ 0\ \ldots$

2nd element of S_2

$S_2 : \ 1\ \ \boxed{1}\ \ 1\ \ 1\ \ 0\ \ 1\ \ 0\ \ldots$

3rd el. of S_3

$S_3 : \ 1\ \ 1\ \ \boxed{0}\ \ 1\ \ 0\ \ 0\ \ 0\ \ldots$

Consider a new element $s \in \{0,1\}^{\mathbb{N}}$ built as follows:

$s : \ \underbrace{1}_{1}\ \ \underbrace{0}_{2}\ \ \underbrace{1}_{3}\ \ \underbrace{\cdots}_{4} \longleftarrow$ n^{th} element of s is 1 if n^{th} element of S_n is 0 and 0 if the n^{th} element of S_n is 1

Clearly $s \notin E$.

Thus, every countable subset of $\{0,1\}^{\mathbb{N}}$ is a proper subset.

Therefore, $\{0,1\}^{\mathbb{N}}$ is uncountable (otherwise, if $\{0,1\}^{\mathbb{N}}$ was countable, then $\{0,1\}^{\mathbb{N}} \subsetneq \{0,1\}^{\mathbb{N}}$.)

$\square$

Note: If we represent real numbers using binary sequences, then the above proves that $\mathbb{R}$ is uncountable. We will give a different proof of this using topology in a later lecture.