Metric spaces

elements $p \in X$ are called "points"

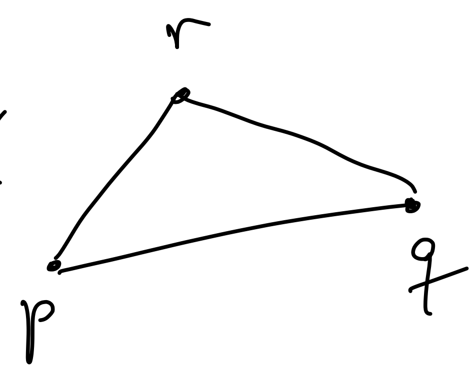
Def: A metric space is a set X with a distance function $d: X \times X \to \mathbb{R}$, i.e., a function s.t.

(1) $d(p,q) > 0$ if $p \neq q$, $d(p,q) = 0 \iff p = q$.

(2) $d(p,q) = d(q,p)$, $\forall p,q \in X$

(3) $d(p,q) \leq d(p,r) + d(r,q)$, $\forall p,q,r \in X$

"triangle inequality".



Example: $X = \mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$

$$d(p,q) = \|p - q\| = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$$

is a metric space ("Euclidean space").

Example: X is any set, $d : X \times X \to \mathbb{R}$

$$d(p,q) = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p \neq q. \end{cases}$$

Check that this "0-1" distance is a dist. function.

Pf:

(1) $d(p,q) = 1 > 0$ if $p \neq q$,

$d(p,q) = 0 \iff p = q$

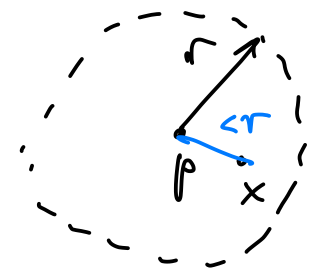
(2) $d(p,q) = d(q,p)$ because $p = q$ and $p \neq q$ are symmetric in p, q

(3) $d(p,q) \leq d(p,r) + d(r,q)$

$p = q$: $d(p,q) = 0$, $d(p,r) + d(r,q) = 2\, d(p,r) = \begin{cases} 0 & p = r \\ 2 & p \neq r \end{cases}$

$p \neq q$: $d(p,q) = 1$, $d(p,r) + d(r,q) = \begin{cases} 1, & \text{if } r = p \neq q \\ 1, & \text{if } p \neq q = r \\ 2, & \text{if } r \neq p \neq q,\ r \neq q. \end{cases}$ $\square$

Def (Metric ball). The (metric) ball of radius $r>0$ centered at $p \in X$ is the set:



$$B_r(p) = \{ x \in X : d(p,x) < r \}.$$

Ex: In the Euclidean space $(\mathbb{R}^n, d)$:

$$B_r(p) = \{ x \in \mathbb{R}^n : \|x-p\| < r \}$$

$B_r(p)$ is the interior of the round sphere of radius r centered at p.

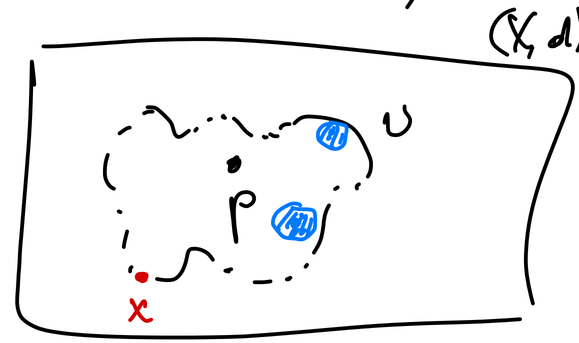
If (X,d) is equipped with the 0-1 distance, then

$$B_r(p) = \{p\}, \text{ if } r \leq 1$$

$$B_r(p) = X, \text{ if } r > 1$$

Def: In a metric space (X,d) we say that:

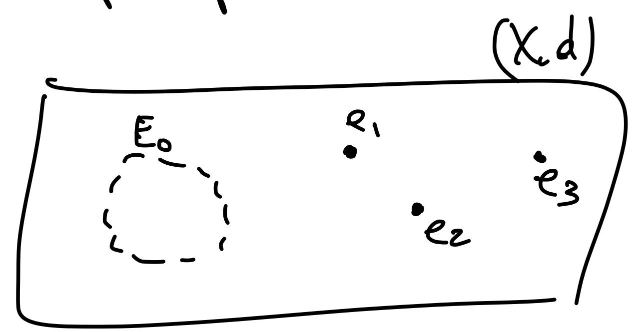
- a (open) <u>neighborhood</u> of $p \in X$ is a subset $U \subset X$ such that $p \in U$ and $\forall q \in U$, $\exists r > 0$, s.t. $B_r(q) \subset U$.



- $p \in X$ is a <u>limit point</u> of $E \subset X$ if <u>every</u> neighborhood of p contains some element $q \in E$, $q \neq p$.

<u>Ex:</u> x "in the boundary" of U is a limit point of U.

- $p \in E$ is an <u>isolated point</u> of $E \subset X$ if p is <u>not</u> a limit point of E.



$E = E_0 \cup \{e_1, e_2, e_3\}$

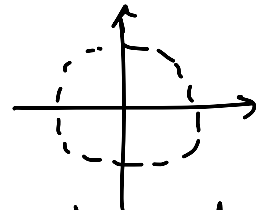
- $E \subset X$ is <u>closed</u> if every limit point of E belongs to E.

- A point p is an <u>interior point</u> of E if there is a neighborhood U of p with $U \subset E$.
- $E \subset X$ is <u>open</u> if every point $x \in E$ is an interior point of E.
- $E \subset X$ is <u>perfect</u> if E is <u>closed</u> and every point of E is a limit point of E.
- $E \subset X$ is <u>bounded</u> if $\exists\, p \in X$ and $\exists\, R > 0$ s.t.

  $E \subset B_R(p)$.
- $E \subset X$ is <u>dense</u> in X if every point of X is a limit point of E or belongs to E.

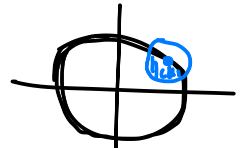
<u>Examples</u>: Determine if the following subsets of $\mathbb{R}^2$ w/ the Euclidean distance are (1) closed; (2) open; (3) perfect; (4) bounded.

A. $B_1(0) = \{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 < 1\}$



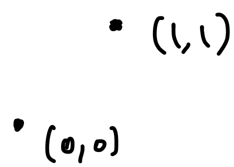
closed: <u>No</u>; open: <u>Yes</u>; perfect: <u>No</u>; bounded: <u>Yes</u>

B. $\overline{B_1(0)} = \{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 \leq 1\}$



closed: <u>Yes</u>, open: <u>No</u>; perfect: <u>Yes</u>, bounded: <u>Yes</u>.

C. $\{(0,0), (1,1)\}$

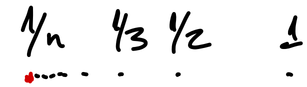


closed: <u>Yes</u>; open: <u>No</u>, perfect: <u>No</u>, bounded: <u>Yes</u>.

D. $\mathbb{Z}$

closed: <u>Yes</u>, open: <u>No</u>; perfect: <u>No</u>; bounded: <u>No</u>

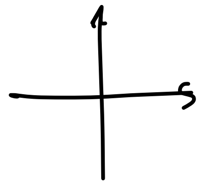
E. $E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$



$0 \notin E$

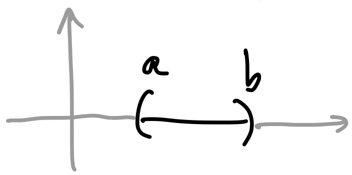
closed: <u>NO</u>, open: <u>NO</u>, perfect: <u>NO</u>; bounded: <u>Yes</u>

F. $\mathbb{R}^2$



closed: <u>Yes</u>, open: <u>Yes</u>; perfect: <u>Yes</u>, bounded: <u>No</u>.

G. (a,b)



closed: <u>NO</u>, open: <u>NO</u> (in $\mathbb{R}^2$); perfect: <u>No</u>; bounded: <u>Yes</u>.

(but: it <u>is open</u> as a subset of $\mathbb{R}$)

<u>Thm</u>: A set E in a metric space (X,d) is <u>open</u> if and only if its complement $E^c = X \setminus E$ is <u>closed</u>.

Pf: First, suppose E^c is closed. Take $x \in E$. Since $x \notin E^c$, x is not a limit point of E^c. Thus there is some neighborhood $U \ni x$ with $U \cap E^c = \phi$, that is, $U \subset E$. This means that $x \in E$ is an interior point of E. Therefore, E is open.

Conversely, suppose E is open. Let $x \in X$ be a limit point of E^c. Then every neighbd. of x contains some point of E^c (other than x itself). So x is not an interior point of E. Thus $x \notin E$, i.e., $x \in E^c$. Therefore E^c is closed. $\square$

Recall: De Morgan's Laws

$$\left(\bigcup_{\alpha \in A} E_\alpha \right)^c = \bigcap_{\alpha \in A} (E_\alpha)^c \quad , \quad \left(\bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} (E_\alpha)^c$$

(not necessarily finite)

Thm. (a) Given any collection $\{G_\alpha\}$ of open sets, the union $\bigcup_{\alpha \in A} G_\alpha$ is open.

(b) Given any collection $\{F_\alpha\}$ of closed sets, the intersection $\bigcap_{\alpha \in A} F_\alpha$ is closed.

(c) If $G_1, \ldots, G_n$ is a finite collection of open sets, then $\bigcap_{i=1}^{n} G_i = G_1 \cap \ldots \cap G_n$ is open

(d) If $F_1, \ldots, F_n$ is a finite collection of closed sets, then $\bigcup_{i=1}^{n} F_i = F_1 \cup \ldots \cup F_n$ is closed.

Pf. (a) $G = \bigcup_{\alpha \in A} G_\alpha$. If $x \in G$, then $\exists \alpha \in A$ s.t. $x \in G_\alpha$. Since G_α is open, x is an interior point of G_α; hence also an interior point of G

Thus, G is open.

(b) $\left(\bigcap_{\alpha\in A} F_\alpha\right)^c = \bigcup_{\alpha\in A} \underbrace{(F_\alpha)^c}_{\text{open}}$ by (a) this is open.

Therefore $\bigcap_{\alpha\in A} F_\alpha$ is closed.

(c) Let $H = \bigcap_{i=1}^{n} G_i$, $x \in H$. Since $x \in G_i$, $\forall i = 1, \dots, n$.
there exist $r_i > 0$ s.t. $B_{r_i}(x) \subset G_i$, $\forall i = 1, \dots, n$.
Let $r = \min_{1\le i\le n} \{r_i\} > 0$. Note $B_r(x) \subset B_{r_i}(x) \subset G_i$,

so $B_r(x) \subset H$. Hence x is an interior point of H,
proving that H is open.

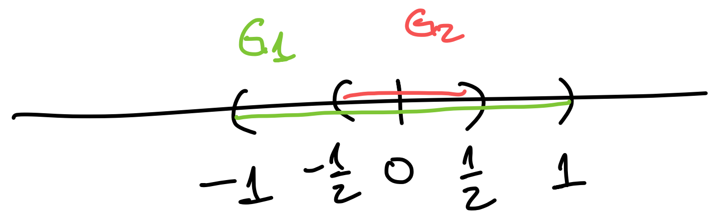
(d) $\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} (F_i)^c$ is open by (c), thus $\bigcup_{i=1}^{n} F_i$ is closed.

$\uparrow$ Open

$\square$

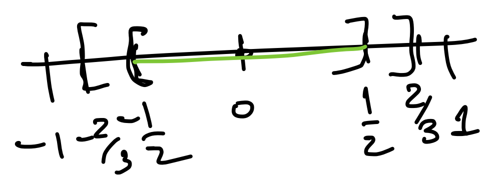
<u>Remark</u>: The condition that A is <u>finite</u> cannot be relaxed in (c) and (d) of Thm. above:

e.g., $G_n := \left(-\frac{1}{n}, \frac{1}{n}\right)$



$$\bigcap_{n\in\mathbb{N}} G_n = \{0\}. \text{ is not open.}$$

Similarly, consider $F_n = \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$



$$\bigcup_{n\in\mathbb{N}} F_n = (-1,1) \text{ is not closed.}$$

Def: The **closure** $\overline{E}$ of a subset E in a metric space is $\overline{E} = E \cup E'$ where E' is the set of limit points of E.

This implies that $\overline{E}$ is the smallest closed set containing E.

Thm. (a) $\overline{E}$ is closed

(b) $E = \overline{E}$ if and only if E is closed

(c) $\overline{E} \subset F$ for every closed set F with $E \subset F$

Pf: (a) If $p \in X$, $p \notin \overline{E}$ then $p \notin E$ and p is not a limit point of E. Thus $\exists U \ni p$ neighbhd. s.t. $U \subset E^c$. Thus $\overline{E}^c$ is open, therefore $\overline{E}$ is closed.

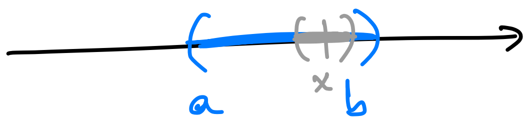
(b) If $E = \overline{E}$, then E is closed by (a). Conversely, if E is closed, then $E' \subset E$ so $\overline{E} = E \cup E' \subset E$.

(c) If F is closed, $E \subset F$, then $E' \subset F' \subset F$.

Then $\overline{E} = E \cup E' \subset F$. $\square$

<u>Openness is relative</u>:

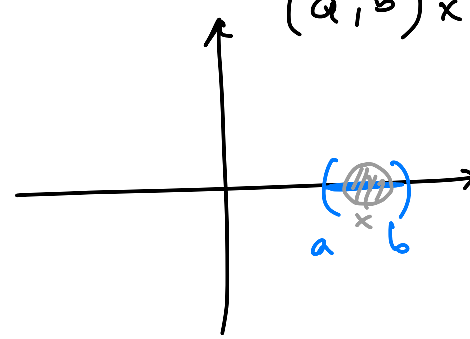
$(a,b) \subset \mathbb{R}$



is <u>open</u>.

v.

$(a,b) \times \{0\} \subset \mathbb{R}^2$



is <u>not open</u>

<u>Note</u>: If $Y \subset (X,d)$ is a subset of the metric space (X,d), then $(Y, d|_{Y \times Y})$ is itself a metric space.

<u>Thm</u>: Suppose $Y \subset X$. A subset $E \subset Y$ is <u>open in Y</u> if and only if $\exists G \subset X$ open in X s.t. $E = G \cap Y$.

Pf: ($\Rightarrow$) Suppose $E \subset Y$ is open in Y. Then $\forall p \in E$ $\exists r_p > 0$ s.t. $\{y \in Y : d(p,y) < r_p\} \subset E$. Let

$$V_p := \{x \in X : d(p,x) < r_p\}, \quad G := \bigcup_{p \in E} V_p$$

Clearly V_p is open in X, and hence so is G.
Since $p \in V_p$, $\forall p \in E$, it follows that $E \subset G \cap Y$.
Moreover, $V_p \cap Y = E$, $\forall p \in E$, so $G \cap Y \subset E$. Therefore $E = G \cap Y$.

Conversely, if $E = G \cap Y$, with $G \subset X$ open in X, then $\forall p \in E$, $\exists V_p \subset G$ neighbd., so $V_p \cap Y \subset E$. Therefore E is open in Y.

$\square$