Name: ANSWERS Lehman ID: ______________

MAT 330/681
Midterm Exam
March 16, 2020

Instructions:
Turn off and put away your cell phone.
Please write your Name and Lehman ID # on the top of this page.
Please sign and date the pledge below to comply with the Code of Academic Integrity.
No consultation material, calculators, or electronic devices are allowed during the exam.
If any question is unclear, raise your hand to ask for clarifications.
You do not need to simplify your answers, unless explicitly required.
The regular amount of time you have to complete the exam is 100 minutes.
You must show all of your work! *No credit will be given for unsupported answers.*
Please try to be as organized, objective, and logical as possible in your answers.

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

My signature below certifies that I complied with the CUNY Academic Integrity Policy and the Lehman College Code of Academic Integrity in completing this examination.

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Signature Date

Problem 1 (10 pts): From a total of 50 kids in an elementary school class, where 26 are boys and 24 are girls, you need to choose 10 boys and 10 girls to participate in a school parade. In how many ways can you choose the kids for the parade?

$$\binom{26}{10} \cdot \binom{24}{10} = \frac{26!}{10!\,16!} \cdot \frac{24!}{10!\,14!} \quad \left(\cong 10^{13}\right)$$

Choose 10 out of 26 boys <u>and</u> Choose 10 out of 24 girls

Problem 2 (10 pts): Assume that the probability that a Category 5 hurricane hits New York City in any given year is 10%. What is the probability of having no Category 5 hurricane hitting New York City for 15 straight years?

$p = \frac{1}{10}$ ← Probability of a Cat. 5 hurricane in a given year

$$P\left(\begin{array}{c}\underline{\text{No}}\text{ Cat. 5 hurricane} \\ \text{in 15 straight years}\end{array}\right) = (1-p)^{15} = \left(\frac{9}{10}\right)^{15} = \frac{9^{15}}{10^{15}}$$

$$\left(\cong 20.59\%\right)$$

Problem 3 (10 pts): Consider the word "COLLEGE".

a) **(5 pts)** How many different words can be written reshuffling the letters in this word?

7 letters
2 E's
2 L's

$$\binom{7}{2,2} = \boxed{\frac{7!}{2!2!}} \left(= 1{,}260\right)$$

b) **(5 pts)** If you pick one of these words at random, what is the probability that the two E's appear together?

$$\text{COLL}\underbrace{(\text{EE})}\text{G}$$

treat as one "combo" letter

6 symbols
2 L's

$$\frac{6!}{2!} \cdot \frac{2!}{2!} = \frac{6!}{2!}$$

$$p = \frac{\frac{6!}{\cancel{2!}}}{\frac{7!}{\cancel{2!}2!}} = \frac{6!\,2!}{7!} = \boxed{\frac{2}{7}}$$

Problem 4 (10 pts): Suppose you have a bag of M&M's with 20 blue M&M's, 15 green M&M's, 35 red M&M's, and 17 yellow M&M's. You are very hungry and eat all M&M's in this bag, but just one at a time (despite being very hungry).

a) **(4 pts)** How many possible orders are there to eat all M&M's in the bag?

20 Blue
15 green
35 Red
17 yellow

87 total

$$\binom{87}{20,15,35,17} = \boxed{\frac{87!}{20!\,15!\,35!\,17!}}$$

(All possible permutations of M&M's)

b) **(4 pts)** If you pick M&M's from the bag in a random order to eat them, what is the probability that you first eat all the red M&M's?

(Permutations of M&M's w/ all red ones first) = (Permutations of all 52 non-red M&M's)

$$= \binom{52}{20,15,17} = \frac{52!}{20!\,15!\,17!}$$

Answ. $$P = \frac{\frac{52!}{20!\,15!\,17!}}{\frac{87!}{20!\,15!\,35!\,17!}} = \boxed{\frac{52!\,35!}{87!}}$$

c) **(2 pts)** If you pick M&M's from the bag in a random order to eat them, what is the probability that the red M&M's are the ones you eat last?

Same as above!

(Permutations of M&M's w/ all red ones last) = (Permutations of all 52 non-red M&M's)

Answ: $$\boxed{P = \frac{52!\,35!}{87!}}$$

Problem 5 (10 pts): According to the United States Centers for Disease Control and Prevention (CDC), among individuals 18–25 years old in the USA, it is reported[1] that:

- 20% have used cannabis in the past month,
- 60% have consumed alcoholic beverages in the past month,
- 10% have smoked cigars in the last month.

Assume using cannabis, consuming alcohol, and smoking cigars are independent but not mutually exclusive events. What is the probability that an individual 18–25 years old either used cannabis or consumed alcohol or smoked cigars in the past month?

$$P(C) = \frac{20}{100} = \frac{1}{5} \leftarrow \text{[Cannabis]}$$

$$P(A) = \frac{60}{100} = \frac{3}{5} \leftarrow \text{[Alcoholic beverages]}$$

$$P(S) = \frac{10}{100} = \frac{1}{10} \leftarrow \text{[smoke cigars]}$$

Inclusion-Exclusion

$$P(C \cup A \cup S) = P(C) + P(A) + P(S) - P(CA) - P(CS) - P(AS) + P(CAS)$$

Indep.

$$= P(C) + P(A) + P(S) - P(A)P(C) - P(C)P(S) - P(A)P(S) + P(C)P(A)P(S)$$

$$= \frac{1}{5} + \frac{3}{5} + \frac{1}{10} - \frac{3}{25} - \frac{1}{50} - \frac{3}{50} + \frac{3}{250}$$

$$= \boxed{\frac{89}{125}} \quad (\cong 71.2\%)$$

[1]Note: these are approximate figures (to simplify computations), from the year 2017. Actual CDC reported figures are: 22.1% used cannabis, 56.3% consumed alcohol, and 9.1% smoked cigars. Reference: `https://www.cdc.gov/nchs/data/hus/2018/020.pdf`

Problem 6 (10 pts): You go to a casino with 200 slot machines, and in 100 of them players win 20% of the time, while in the other 100 of them players win 40% of the time. Suppose you choose one slot machine at random and play 10 times, winning [redacted] exactly 4 [redacted] of [redacted] them. What is the probability that you were playing in a machine where players win 40% of the time?

H = you played the slot machine that pays 40% of the time (hypoth.)

E = win 4 out of 6 times in a slot machine (evidence)

$$P(H|E) = \frac{P(E|H)\,P(H)}{P(E|H)\,P(H) + P(E|H^c)\,P(H^c)}$$

$$= \boxed{\frac{\binom{10}{4}(0.4)^4(0.6)^6}{\binom{10}{4}(0.4)^4(0.6)^6 + \binom{10}{4}(0.2)^4(0.8)^6}}$$

$$= \frac{1}{1 + \frac{2^4\,8^6}{4^4\,6^6}} = \frac{1}{1 + \frac{2^4 \cdot 2^{18}}{2^8\,2^6\,3^6}} = \frac{1}{1 + \frac{2^8}{3^6}}$$

$$= \frac{3^6}{3^6 + 2^8} \quad (\cong 74.01\,\%)$$

Problem 7 (10 pts): In the Gambler's Ruin Problem, if player A starts with $\$i$ and has probability $0 < p \leq 1$ of winning each round of a game against player B, who starts with $\$(N-i)$, then the probability that A ends up winning the entire jackpot $\$N$ (hence ending the game as the overall winner) is shown to be:

$$P(\text{A wins jackpot of }\$N) = \begin{cases} \dfrac{1-\left(\frac{1-p}{p}\right)^i}{1-\left(\frac{1-p}{p}\right)^N}, & \text{if } p \neq \dfrac{1}{2} \\[2ex] \dfrac{i}{N}, & \text{if } p = \dfrac{1}{2}. \end{cases}$$

a) **(6pts)** In the limit as the total jackpot $\$N$ approaches $+\infty$, what is the probability that A wins the entire jackpot?

$$L = \lim_{N\to\infty}\left(\frac{1-p}{p}\right)^N = \begin{cases} 0 & \text{if } \frac{1-p}{p} < 1, \text{ i.e., if } p > \frac{1}{2} \\ +\infty & \text{if } \frac{1-p}{p} > 1, \text{ ie. if } p < \frac{1}{2} \end{cases}$$

of course, $\lim_{N\to\infty} \frac{i}{N} = 0$ (if $p = \frac{1}{2}$)

So,

$$\boxed{\lim_{N\to\infty} P(\text{A wins}) = \begin{cases} 1-\left(\frac{1-p}{p}\right)^i & \text{if } p > \frac{1}{2} \\ 0 & \text{if } p \leq \frac{1}{2} \end{cases}}$$

b) **(4pts)** If $p = \frac{2}{3}$, simplify the above result into a fraction $\frac{m}{n}$ where m, n are integers. (use $i = 10$)

$$p = \frac{2}{3} \Rightarrow \frac{1-p}{p} = \frac{1/3}{2/3} = \frac{1}{2}.$$

In this case, since $p > \frac{1}{2}$, we have

$$\lim_{N\to\infty} P(\text{A wins}) = 1 - \left(\frac{1}{2}\right)^{10} = 1 - \frac{1}{2^{10}} = \frac{2^{10}-1}{2^{10}} = \boxed{\frac{1023}{1024}}$$

Problem 8 (10 pts): According to the US Department of Transportation and National Highway Traffic Safety Administration, there were (approximately[2]) 6,000,000 car crashes in the USA in 2017. There were 30,000 fatal crashes, and, in 8,000 of these, the driver was over the speed limit. Knowing that 22% of drivers always obey the speed limit, find the probability that a crash was fatal given that the driver was under the speed limit.

Simplify your answer into a fraction of the form $\frac{m}{n}$ where m, n are integers.

(using units of thousands everywhere)

- 6,000 crashes
- 30 fatal crashes
- 8 fatal crashes and speeding, (so 22 fatal and not speeding)

F = crash is fatal

S = speeding , S^c = obeying speed limit.

$$P(F) = \frac{30}{6{,}000} = \frac{3}{600} = \frac{1}{200}$$

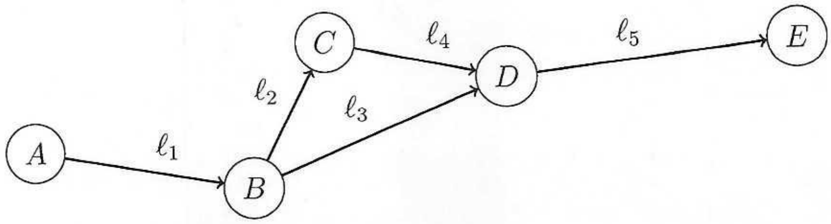
$$P(S^c) = \frac{22}{100}$$

$$P(F \mid S^c) = \frac{P(FS^c)}{P(S^c)} = \frac{22/6000}{22/100} = \frac{100}{6000} = \boxed{\frac{1}{60}}$$

$(\simeq 1.6\%)$

[2] Data from US DOT and NHTSA in 2017: https://www.nhtsa.gov/risky-driving/

Problem 9 (10 pts): Suppose that a communication network has nodes A, B, C, D, E and links ℓ_1, ℓ_2, ℓ_3, ℓ_4, ℓ_5, as in the diagram below:



The probability that each of the above links ℓ_i is available at any given time is p, and the availabilities of links are independent from one another. In order for a message to be successfully sent from A to E, there must be at least one path of available links from A to E. What is the probability that a message is successfully sent from A to E?

2 Paths possible: $\ell_1 \ell_2 \ell_4 \ell_5$ or $\ell_1 \ell_3 \ell_5$

$P(\ell_i) = p$ ← prob. that link ℓ_i is available.

$$P(\ell_1 \ell_2 \ell_4 \ell_5 \cup \ell_1 \ell_3 \ell_5) = P(\ell_1 \ell_2 \ell_4 \ell_5) + P(\ell_1 \ell_3 \ell_5) - P(\ell_1 \ell_2 \ell_3 \ell_4 \ell_5)$$

$$= P(\ell_1)P(\ell_2)P(\ell_4)P(\ell_5) + P(\ell_1)P(\ell_3)P(\ell_5) - P(\ell_1)P(\ell_2)P(\ell_3)P(\ell_4)P(\ell_5)$$

$$= \boxed{p^4 + p^3 - p^5}$$

Problem 10 (10 pts): When you enter the NYC subway by swiping your MTA card at a turnstyle, you pay \$2.75. The evasion fine (for being caught riding the subway without swiping your card) is \$100. Suppose that the MTA police catches only 1 in 20 people that ride the subway without swiping their card. Use the expected value of a random variable to decide whether, over a long period of time, it is cheaper to swipe your card at the turnstyle or not.

Hint: Consider the random variable X which is the amount you pay when you ride the subway without swiping your card at the turnstyle.

X = amount you pay when you ride without swiping

$P(X = 100) = \frac{1}{20}$ (getting caught)

$P(X = 0) = \frac{19}{20}$ (not getting caught)

$$E(X) = 0 \cdot \frac{19}{20} + 100 \cdot \frac{1}{20} = \frac{10}{2} = \boxed{5}$$

Expected amount you pay when you ride without swiping is \$5, which is higher than the amount you pay when swiping (\$2.75).

Thus, over a long period of time, it is cheaper to swipe.

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