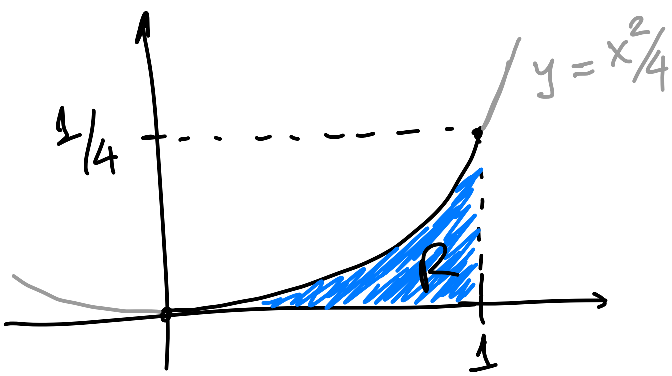
MAT330 Solution to HW 10

1. $X, Y \sim \mathrm{Unif}([0,1])$, independent.

$a^2 + Xa + Y = 0$ has 2 distinct real solutions $a_1, a_2 \in \mathbb{R}$ if and only if $\Delta = X^2 - 4Y > 0$.

(in which case $a_1 = \dfrac{-X + \sqrt{X^2 - 4Y}}{2}$, $a_2 = \dfrac{-X - \sqrt{X^2-4Y}}{2}$)

$$P(X^2 - 4Y > 0) = P\left(Y < X^2/4\right) = \frac{\mathrm{Area}(R)}{\mathrm{Area}([0,1]^2)}$$



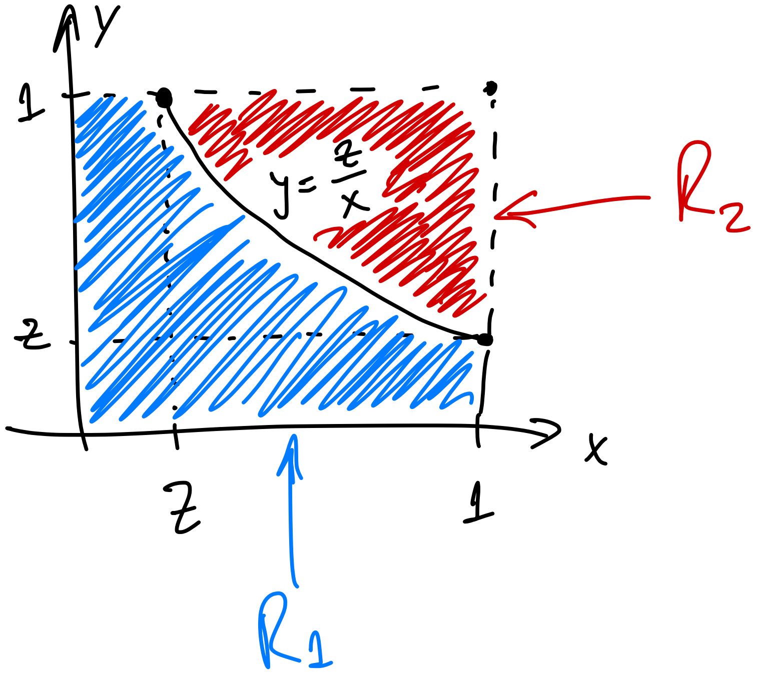
$$= \int_0^1 \frac{x^2}{4}\,dx = \frac{x^3}{12}\Big|_0^1 = \boxed{\frac{1}{12}}$$

2. $X, Y \sim \mathrm{Unif}([0,1])$, independent.

a) $E(XY) \overset{\text{indep.}}{=} E(X)\,E(Y) = \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4}}$

b) $Z = XY$ also takes values only on $[0,1]$.

$$F_Z(z) = P(Z \le z) = P(XY \le z) = P\left(Y \le \frac{z}{X}\right) =$$



$$= \frac{\text{Area}(R_1)}{\text{Area}([0,1]^2)}$$

$$= 1 - \text{Area}(R_2)$$

$$= 1 - \int_z^1 \int_{z/x}^1 1 \, dy \, dx$$

$$= 1 - \int_z^1 \left(1 - \frac{z}{x}\right) dx$$

$$= 1 - \left(x - z\ln x\right)\Big|_z^1 = 1 - \left(1 - z\cancelto{0}{\ln 1} - z + z\ln z\right)$$

$$= z - z\ln z.$$

Thus $f_Z(z) = \dfrac{d}{dz} F_Z(z) = \dfrac{d}{dz}\left(z - z\ln z\right) =$

$$= 1 - \ln z - z \cdot \frac{1}{z} = -\ln z = \boxed{\ln \frac{1}{z}}$$

c) $E(Z) = \displaystyle\int_0^1 z \cdot \ln \frac{1}{z} \, dz = -\int_0^1 z \ln z \, dz =$ ← improper integral

$$= -\lim_{a \downarrow 0+} \int_a^1 z \ln z \, dz \overset{\text{parts}}{=} -\lim_{a \downarrow 0+} \left(\frac{z^2}{2} \ln z\right)\Big|_a^1 - \int_a^1 \frac{z^2}{2} \cdot \frac{1}{z} \, dz$$

$$= -\lim_{a \downarrow 0_+} \left(\frac{1}{2} \cancel{\ln 1}_{0} - \frac{a^2}{2} \ln a \right) - \int_a^1 \frac{z}{2}\, dz$$

$$= + \lim_{a \downarrow 0_+} \frac{a^2}{2} \ln a + \left. \frac{z^2}{4} \right|_a^1$$

$$= \lim_{a \downarrow 0_+} \underbrace{\frac{a^2 \ln a}{2}}_{\to 0 \text{ (L'Hospital)}} + \frac{1}{4} - \underbrace{\frac{a^2}{4}}_{\to 0} = \boxed{\frac{1}{4}}.$$

Matches answer found in part a).

$$\left[\lim_{a \to 0} a^2 \ln a = \lim_{a \to 0} \frac{\ln a}{1/a^2} \overset{\text{L'H}}{=} \lim_{a \to 0} \frac{1/a}{-2/a^3} = \lim_{a \to 0} -\frac{a}{2} = 0. \right]$$