Solution to HW 11

#1. $X \sim \text{Uniform}([0,1]) \implies f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

a) $M(t) = E(e^{tX}) = \int_0^1 e^{tx}\,dx = \frac{e^{tx}}{t}\Big|_0^1 = \frac{e^t}{t} - \frac{1}{t} = \boxed{\frac{e^t - 1}{t}}$

b) $M'(t) = \frac{e^t t - (e^t - 1)}{t^2} = \frac{e^t(t-1) + 1}{t^2}$

$$M'(0) = \lim_{t\to 0} \frac{e^t(t-1)+1}{t^2} \overset{\text{L'H.}}{=} \lim_{t\to 0} \frac{e^t(t-1) + e^t}{2t} = \lim_{t\to 0} \frac{\cancel{t}e^t}{2\cancel{t}} = \boxed{\frac{1}{2}}$$

$$M''(t) = \frac{d}{dt} \frac{e^t(t-1)+1}{t^2} = \frac{(t^2-2t+2)e^t - 2}{t^3}$$

$$M''(0) = \lim_{t\to 0} \frac{(t^2-2t+2)e^t - 2}{t^3} = \boxed{\frac{1}{3}}$$

$$M'''(t) = \frac{d}{dt} \frac{(t^2-2t+2)e^t - 2}{t^3} = \frac{(t^3-3t^2+6t-6)e^t + 6}{t^4}$$

$$M'''(0) = \lim_{t\to 0} \frac{(t^3-3t^2+6t-6)e^t + 6}{t^4} = \boxed{\frac{1}{4}}$$

(in general, $M^{(n)}(0) = \frac{1}{n+1}$. Try to prove it!)

c) $\text{Var}(X) = E(X^2) - E(X)^2 = M''(0) - M'(0)^2 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$

#2 $Y \sim \text{Poisson}(5) \Rightarrow p_Y(y) = \dfrac{5^y e^{-5}}{y!}, \quad y \geq 0.$

a) $$M(t) = E(e^{tY}) = \sum_{y=0}^{+\infty} e^{ty} \frac{5^y e^{-5}}{y!} = \frac{1}{e^5} \sum_{y=0}^{+\infty} \frac{e^{ty + y\ln 5}}{y!}$$

$$= \frac{1}{e^5} \sum_{y=0}^{+\infty} \frac{e^{y(t+\ln 5)}}{y!} = \frac{1}{e^5} \sum_{y=0}^{+\infty} \frac{\left[e^{t+\ln 5}\right]^y}{y!} = \frac{1}{e^5} e^{e^{t+\ln 5}}$$

$$\left(\sum_{y=0}^{+\infty} \frac{a^y}{y!} = e^a \right)$$

$$= e^{-5} \cdot e^{e^t \cdot e^{\ln 5}} = \boxed{e^{5e^t - 5}}$$

b) $$M'(t) = 5e^{5e^t + t - 5}$$

$$M'(0) = 5e^{5-5} = \boxed{5}$$

$$M''(t) = 5e^{5e^t + t - 5}\left(5e^t + 1\right)$$

$$M''(0) = 5 \cdot (5+1) = \boxed{30}$$

$$M'''(t) = 5e^{5e^t + t - 5}\left(25e^{2t} + 15e^t + 1\right)$$

$$M'''(0) = 5 \cdot (25 + 15 + 1) = 5 \cdot 41 = \boxed{205}$$

Recall that if $Y \sim \text{Poisson}(\lambda)$, then;
$E(Y) = \lambda$
$Var(Y) = \lambda$

c) $$Var(Y) = E(Y^2) - E(Y)^2 = 30 - 5^2 = \boxed{5}$$

#3. If X, Y are independent, then

$$M_{X+Y}(t) = E\left(e^{t(X+Y)}\right) = E\left(e^{tX} \cdot e^{tY}\right) = E\left(e^{tX}\right) E\left(e^{tY}\right)$$

$$= M_X(t)\, M_Y(t) = \boxed{\left(\frac{e^t - 1}{t}\right) \cdot e^{5e^t - 5}}$$