MAT 330/681

Solutions to HW4

K_n = student knows material of question n

C_n = student gets correct answer on question n

$$P(C_n | K_n) = 1, \qquad P(C_n | K_n^c) = \frac{1}{5} \quad \text{(guess at random)}$$

a) $$P(K_n | C_n) = \frac{P(K_n C_n)}{P(C_n)} = \frac{P(C_n|K_n)P(K_n)}{P(C_n|K_n)P(K_n) + P(C_n|K_n^c)P(K_n^c)}$$

$$= \frac{p_n}{p_n + \frac{1}{5}(1-p_n)} = \frac{5p_n}{4p_n+1}.$$

so

$$P(\underbrace{K_1K_2 \cdots K_6}_{\text{know everything}} \,|\, \underbrace{C_1C_2\cdots C_6}_{\text{got all answers correct}}) \overset{\text{(indep.)}}{=} P(K_1|C_1)P(K_2|C_2)\cdots P(K_6|C_6)$$

GIVEN THAT

$$= \boxed{\prod_{n=1}^{6} \frac{5p_n}{4p_n+1}}$$

b) $$P(K_n^c C_n) = P(C_n|K_n^c)P(K_n^c) = \frac{1}{5}(1-p_n)$$

$$P(\underbrace{K_1^c \cdots K_6^c}_{\text{don't know anything}} \underbrace{C_1 \cdots C_6}_{\text{got all answers correct}}) = P(K_1^c C_1 \cdot K_2^c C_2 \cdots K_6^c C_6)$$

AND

$$\overset{\text{(indep)}}{=} P(K_1^c C_1) \cdots P(K_6^c C_6)$$

$$= \boxed{\prod_{n=1}^{6} \frac{1-p_n}{5}}$$

c) $p_n = \frac{1}{2}$, $\forall n = 1, \ldots, 6$

a) $\prod_{n=1}^{6} \frac{5/2}{4/2+1} = \left(\frac{5}{6}\right)^6$ $(\cong 33.49\%)$

b) $\prod_{n=1}^{6} \frac{1}{10} = \frac{1}{10^6}$ $(\cong 0.0001\%)$

Bonus part. d) $1 - P(K_1 \ldots K_6 | C_1 \ldots C_6) = \boxed{1 - \prod_{n=1}^{6} \frac{5p_n}{4p_n+1}}$

or: $P\Big(\underbrace{\bigcup_{n=1}^{6} K_n^c}_{\text{did not know at least 1 question (hence guessed)}} \Big| \underbrace{C_1 \ldots C_6}_{\text{got all answers correct}}\Big) = \frac{P\left(\left(\bigcup_{n=1}^{6} K_n^c\right) C_1 \ldots C_6\right)}{P(C_1 \ldots C_6)}$

GIVEN THAT

$$= \frac{P\left(\bigcup_{n=1}^{6} (K_n^c C_1 \ldots C_6)\right)}{P(C_1 \ldots C_6)}$$

$$P(C_n) = P(C_n|K_n)P(K_n) + P(C_n|K_n^c)P(K_n^c) = p_n + \frac{1}{5}(1-p_n)$$

$$\Rightarrow P(C_1 \ldots C_6) = \prod_{n=1}^{6} \left(p_n + \frac{1-p_n}{5}\right)$$

(Note: If $p_n = \frac{1}{2}$, then $P(C_1 \ldots C_6) = \left(\frac{3}{5}\right)^6$)

$$P\left(\bigcup_{n=1}^{6} K_n^c C_1 \cdots C_6\right) = \sum_{r=1}^{6} (-1)^{r+1} \sum_{i_1<\cdots<i_r} P\left(K_{i_1}^c \cdots K_{i_r}^c C_1 \cdots C_6\right)$$

inclusion-exclusion principle

$$= \sum_{r=1}^{6} (-1)^{r+1} \sum_{i_1<\cdots<i_r} P\left(K_{i_1}^c C_{i_1} \cdots K_{i_r}^c C_{i_r} \cdot \underbrace{C_{\hat{i_j}}}\right)$$

only the ones that do not appear among $i_1, \ldots, i_r$

$$= \sum_{r=1}^{6} (-1)^{r+1} \sum_{i_1<\cdots<i_r} P\left(K_{i_1}^c C_{i_1}\right) \cdots P\left(K_{i_r}^c C_{i_r}\right) P\left(C_{\hat{i_j}}\right)$$

$$= \sum_{r=1}^{6} (-1)^{r+1} \sum_{i_1<\cdots<i_r} \left(\prod_{j=1}^{r} \frac{1-p_{i_j}}{5}\right)\left(\prod_{\hat{i_j}} p_{\hat{i_j}} + \frac{1-p_{\hat{i_j}}}{5}\right)$$

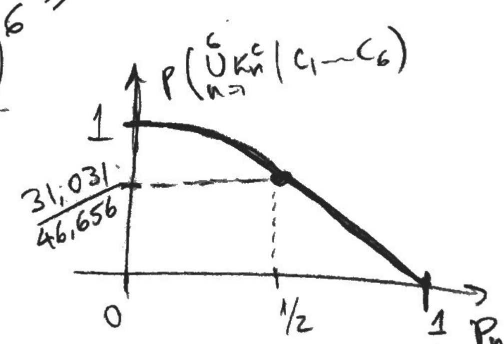
If $p_n = \frac{1}{2}, \forall n$:

$$= \sum_{r=1}^{6} (-1)^{r+1} \binom{6}{r} \left(\frac{1}{10}\right)^{r} \left(\frac{3}{5}\right)^{6-r} = \frac{31{,}031}{10^6}$$

So $$\underbrace{P\left(\bigcup_{n=1}^{6} K_n^c \,\middle|\, C_1 \cdots C_6\right)} = \frac{31{,}031}{10^6} \cdot \frac{5^6}{3^6} = \frac{31{,}031}{46{,}656} \cong 66.51\,\%$$

guessing at least 1 answer given that got a perfect score. assuming prob. of having learnt each of the 6 questions is 50%

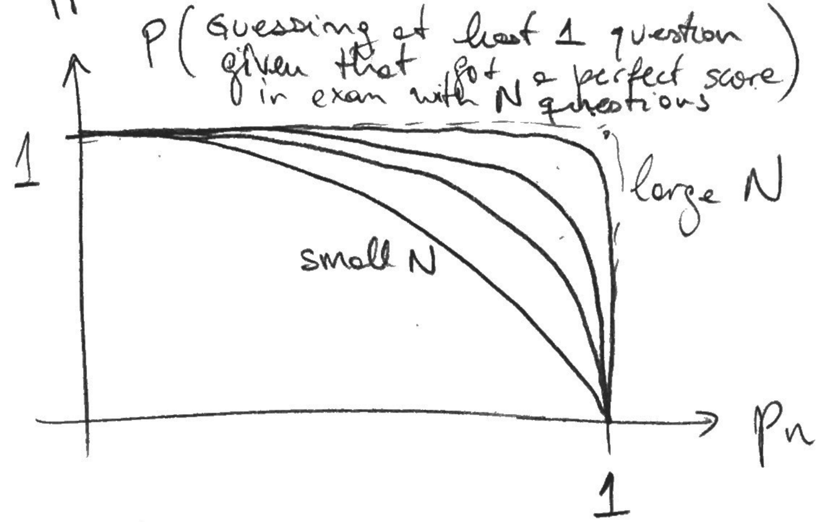
$= $ or $\underline{1 - \left(\frac{5}{6}\right)^6}$



Of course, if p_n was <u>larger</u>, then the above answer would be smaller!

An interesting observation:

Suppose we let the number N of questions go to ∞.



$$1 - \left(\frac{5p_n}{4p_n + 1}\right)^N$$

$$\|$$

$$P\left(\begin{array}{c}\text{Guessing at least 1 question}\\ \text{given that got a perfect score}\\ \text{in exam with N questions}\end{array}\right) = \frac{\sum_{r=1}^{N} (-1)^{r+1} \binom{N}{r} \left(\frac{1-p_n}{5}\right)^r \left(p_n + \frac{1-p_n}{5}\right)^{N-r}}{\left(p_n + \frac{1-p_n}{5}\right)^N}$$

As $N \nearrow +\infty$, the above function converges to

$$\lim_{N\to\infty} P\left(\begin{array}{c}\text{Guessing at least 1 question}\\ \text{given that got a perfect score}\\ \text{in exam with N questions}\end{array}\right) = \begin{cases} 1 & \text{if } p_n < 1 \\ 0, & \text{if } p_n = 1 \end{cases}$$

So, unless you really know everything, you (almost) surely had to guess at least 1 question if you got a perfect score, provided there were enough questions in the exam! :)