<u>Recap:</u> $E(X) = \sum_x x \cdot \underbrace{P(X=x)}_{} = \sum_x x \cdot p(x) = \mu$ mean of X

(X is a discrete random var.) $p(x)$ Probability mass density

$g: \mathbb{R} \to \mathbb{R}$ function

$$E(g(X)) = \sum_x g(x) P(X=x) = \sum_x g(x) p(x)$$

<u>Def:</u> Variance of a (discrete) random variable X is

$$\boxed{\mathrm{Var}(X) = E\left((X-\mu)^2\right) \text{ where } \mu = E(X).}$$

<u>Prop:</u> $\boxed{\mathrm{Var}(X) = E(X^2) - E(X)^2 = E(X^2) - \mu^2}$

<u>Pf:</u> $\mathrm{Var}(X) = E\left((X-\mu)^2\right) = \sum_x (x-\mu)^2 p(x)$

$g(x) = (x-\mu)^2$

$$= \sum_x (x^2 - 2x\mu + \mu^2) p(x) = \sum_x x^2 p(x) - 2\mu x p(x) + \mu^2 p(x)$$

$$= \underbrace{\sum_x x^2 p(x)}_{} - 2\mu \underbrace{\sum_x x p(x)}_{\mu} + \mu^2 \underbrace{\sum_x p(x)}_{1}$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2 \qquad \square$$

Ex: X = result of rolling 1 die

X	1	2	3	...	6
$p(x)$	$1/6$	$1/6$	$1/6$	...	$1/6$

$$E(X) = \sum_x x\,p(x) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + \cdots + 6\left(\frac{1}{6}\right)$$

$$= \frac{1+2+3+4+5+6}{6} = \boxed{\frac{7}{2}}$$

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \sum_x x^2 p(x) = 1^2\frac{1}{6} + 2^2\frac{1}{6} + 3^2\frac{1}{6} + \cdots + 6^2\frac{1}{6}$$

$$= \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = \frac{91}{6}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \boxed{\frac{35}{12}}$$

Interpretation: $Var(X)$ measures how concentrated or spread out the values that X assumes are.

Note: $Var(X) \geq 0$

b/c: $$Var(X) = E\left((X-\mu)^2\right) = \sum_x \underbrace{(x-\mu)^2}_{\geq 0} \underbrace{p(x)}_{\geq 0} \geq 0.$$

Def: Standard deviation of X is

$$\sigma_X = \sqrt{Var(X)} \geq 0$$

Properties of Expected Value and Variance:

X, Y (discrete) random variables, $a, b \in \mathbb{R}$

1. $E(aX+bY) = aE(X) + bE(Y)$ linearity

2. $Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X,Y)$.

where $Cov(X,Y) = E\big((X-E(X))(Y-E(Y))\big)$ is the covariance of X and Y.

(In particular, $Var(aX+b) = a^2 Var(X)$.)

<u>Prop:</u> $Cov(X,Y) = E(XY) - E(X)E(Y)$

<u>Pf:</u> $Cov(X,Y) = E\big((X-E(X))(Y-E(Y))\big)$

$$= E\big(XY - XE(Y) - YE(X) + E(X)E(Y)\big)$$

linearity of $E(\cdot)$

$$= E(XY) - E(XE(Y)) - E(YE(X)) + E(X)E(Y)$$

$$= E(XY) - E(X)E(Y) - E(X)\cancel{E(Y)} + E(X)\cancel{E(Y)}$$

$$= E(XY) - E(X)E(Y).$$

<u>Interpretation:</u> $Cov(X,Y)$ is a measure of joint variability of X and Y.

Correlation:

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

$\boxed{\begin{array}{l}\underline{\text{Independent Random Variables:}} \\ \underline{\text{Def}}\text{: } X, Y \text{ are indep. if the events } X \in A \text{ and} \\ Y \in B \text{ are independent events for all sets } A, B \text{, i.e.} \\ \quad P(X \in A \text{ and } Y \in B) = P(X \in A)\, P(Y \in B)\end{array}}$

In particular, if X and Y are <u>indep.</u> then

$$E(XY) = E(X)E(Y)$$

In this case,

$$\begin{aligned}\mathrm{Cov}(X,Y) &= E(XY) - E(X)E(Y) \\ &= E(X)E(Y) - E(X)E(Y) \\ &= 0.\end{aligned}$$

In particular, $\rho_{X,Y} = 0$.