MAT 330 Lecture 18 4/1/2020

Poisson Random Variables ⟵ Use this on HW7 #2

A discrete random variable X that assumes values $i = 0, 1, 2, 3, 4, \ldots$ is a Poisson with parameter $\lambda > 0$ if

$$P(X=i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \qquad i = 0, 1, 2, 3, \ldots$$

Write this as: $X \sim \text{Poisson}(\lambda)$

Verify that the above is a probability distribution:

$$\sum_{i=0}^{+\infty} P(X=i) = \sum_{i=0}^{+\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = e^{-\lambda} \underbrace{\sum_{i=0}^{+\infty} \frac{\lambda^i}{i!}}_{= e^{\lambda}} = 1$$

$$\boxed{e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}}$$

Recall:

$$\underbrace{\sum_{n=0}^{+\infty} \frac{n x^n}{n!}}_{\frac{0.x^0}{0!} + \frac{1.x^1}{1!} + \cdots} = \underbrace{\sum_{n=1}^{+\infty} \frac{x^n}{(n-1)!}}_{\frac{x^1}{0!} + \cdots} = \sum_{n=1}^{+\infty} \frac{x . x^{n-1}}{(n-1)!} = x \underbrace{\sum_{n=1}^{+\infty} \frac{x^{n-1}}{(n-1)!}}_{= e^x}$$

$$= x \cdot e^x$$

Lemma: $\displaystyle\sum_{n=0}^{+\infty} \frac{x^n}{n!} = e^x, \qquad \sum_{n=0}^{+\infty} \frac{n x^n}{n!} = x e^x$

Using this, we can compute:

$$E(X) = \sum_{i=0}^{+\infty} i\, p(i) = \sum_{i=0}^{+\infty} i \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda} \underbrace{\sum_{i=0}^{+\infty} \frac{i\lambda^i}{i!}}_{\lambda e^{\lambda}}$$

$$= \cancel{e^{-\lambda}} \cdot \lambda \cdot \cancel{e^{\lambda}} = \boxed{\lambda}$$

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \sum_{i=0}^{+\infty} i^2 p(i) = \sum_{i=0}^{+\infty} i^2 \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{+\infty} \frac{i^2\lambda^i}{i!} =$$

$$= e^{-\lambda} \sum_{i=1}^{+\infty} \frac{i\lambda^i}{(i-1)!} \overset{j=i-1}{=} e^{-\lambda} \sum_{j=0}^{+\infty} \frac{(j+1)\lambda^{j+1}}{j!} =$$

$$= e^{-\lambda} \left(\underbrace{\sum_{j=0}^{+\infty} \frac{j \boxed{\lambda^{j+1}}}{j!}}_{\lambda \cdot \lambda e^{\lambda}} + \underbrace{\sum_{j=0}^{+\infty} \frac{1 \cdot \boxed{\lambda^{j+1}}}{j!}}_{\lambda e^{\lambda}} \right) \qquad \boxed{\lambda^{j+1}} = \lambda^j \cdot \lambda$$

$$= e^{-\lambda}\left(\lambda^2 e^{\lambda} + \lambda e^{\lambda}\right) = \lambda^2 + \lambda.$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

$$= \lambda^2 + \lambda - (\lambda)^2 = \lambda^2 + \lambda - \lambda^2 = \boxed{\lambda}$$

$$\boxed{\begin{array}{l} \underline{\text{Summary}}: \; X \sim \text{Poisson}(\lambda) \\ E(X) = \lambda \\ \mathrm{Var}(X) = \lambda \end{array}}$$

A: Why study <u>Poisson</u> random variables?

Q: They provide very good approximations to Binomial random variables with parameters (n, p) if n is <u>large</u> and np is <u>moderate</u>.

$Y \sim \text{Binomial}(n, p)$ $\qquad P(Y = i) = \binom{n}{i} p^i (1-p)^{n-i}$

$\left(\begin{array}{l} \text{Very large } n, \text{ set } \lambda = n \cdot p, \\ \text{and } X \sim \text{Poisson}(\lambda). \end{array}\right)$ $\qquad \cong e^{-\lambda} \dfrac{\lambda^i}{i!} = P(X = i)$

$$P(Y = i) = \binom{n}{i} p^i (1-p)^{n-i} = \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} = \frac{n!}{i!(n-i)!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

$$= \underbrace{\frac{n(n-1)\ldots(n-i+1)}{n^i}}_{\approx 1} \underbrace{\frac{\lambda^i}{i!}}_{\frac{\lambda^i}{i!}} \frac{\overbrace{\left(1 - \frac{\lambda}{n}\right)^n}^{\approx e^{-\lambda}}}{\underbrace{\left(1 - \frac{\lambda}{n}\right)^i}_{\approx 1}} \approx e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Example: Suppose the number of typographical errors on a single page of a book has Poisson distr. w/ parameter $\lambda = \frac{1}{2}$. What is the probability that

a) A given page contains at least 1 error?

$$X = \#\left(\begin{matrix}\text{errors in}\\ \text{a page}\end{matrix}\right) \sim \text{Poisson}\left(\tfrac{1}{2}\right) \Rightarrow P(X=i) = \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^i}{i!}$$

$$P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$$

$$= 1 - \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^0}{0!} = 1 - e^{-\frac{1}{2}} = 1 - \frac{1}{\sqrt{e}} \cong 0.393$$

$$(\cong 39.3\%)$$

b) A given page contains no more than 4 errors?

$$P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= \frac{e^{-\frac{1}{2}}\underbrace{\left(\frac{1}{2}\right)^0}_{=1}}{0!} + \frac{e^{-\frac{1}{2}}\underbrace{\left(\frac{1}{2}\right)^1}_{=\frac{1}{2}}}{1!} + \frac{e^{-\frac{1}{2}}\underbrace{\left(\frac{1}{2}\right)^2}_{=\frac{1}{4}}}{2!} + \frac{e^{-\frac{1}{2}}\underbrace{\left(\frac{1}{2}\right)^3}_{=\frac{1}{8}}}{3!} + \frac{e^{-\frac{1}{2}}\underbrace{\left(\frac{1}{2}\right)^4}_{=\frac{1}{16}}}{4!}$$

$$= e^{-\frac{1}{2}}\left(1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384}\right) = e^{-\frac{1}{2}}\,\frac{211}{128}$$

$$= \frac{211}{128\sqrt{e}} \cong 0.9998 \; (\cong 99.98\%)$$