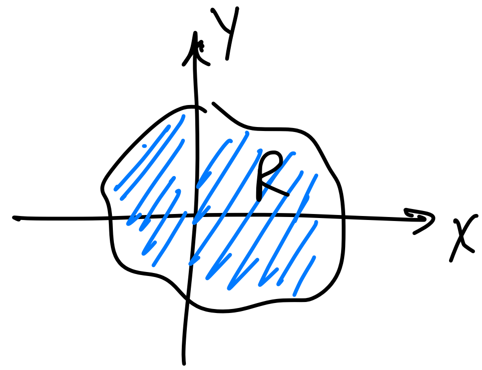
Recall: X, Y jointly distributed

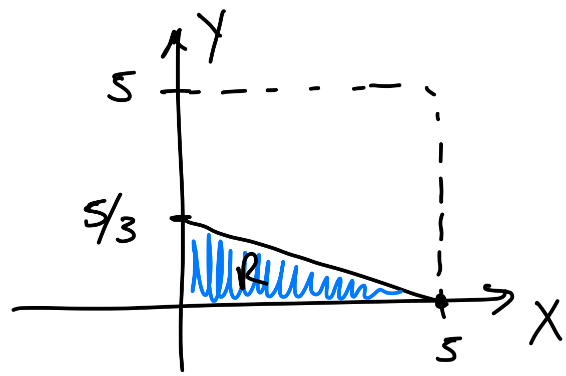
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$P\big((X,Y)\in R\big) = \iint_R f_{X,Y}(x,y)\,dA$$



Examples: $X, Y \sim \mathrm{Unif}([0,5])$

$$P(X+3Y<5) = ?$$



$$R = \{(x,y)\in[0,5]^2,\ x+3y<5\}$$
$$= \{(x,y)\in[0,5]^2,\ y<\tfrac{5}{3}-\tfrac{x}{3}\}$$

$$f_{X,Y} = \frac{1}{25}$$

$$P(X+3Y<5) = P\big((X,Y)\in R\big) = \iint_R \frac{1}{25}\,dA = \frac{1}{25}\iint_R dA$$

$$= \frac{1}{25}\,\mathrm{Area}(R) = \frac{1}{\cancel{25}}\left(\frac{\cancel{5}}{3}\cdot\cancel{5}\cdot\frac{1}{2}\right) = \boxed{\frac{1}{6}}$$

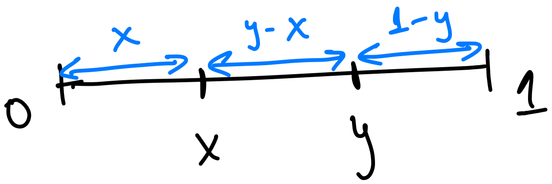
Note: $$\iint_R \frac{1}{25}\,dA = \int_0^5\int_0^{\frac{5}{3}-\frac{x}{3}} \frac{1}{25}\,dy\,dx$$

$$= \int_0^5 \frac{y}{25}\Big|_0^{\frac{5}{3}-\frac{x}{3}} dx = \int_0^5 \frac{1}{25}\left(\frac{5}{3}-\frac{x}{3}\right)dx$$

$$= \int_0^5 \frac{1}{15}-\frac{x}{75}\,dx = \left(\frac{x}{15}-\frac{x^2}{150}\right)\Big|_0^5 = \frac{1}{3}-\frac{1}{6} = \boxed{\frac{1}{6}}$$

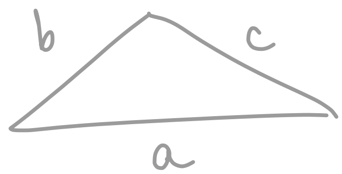
A very classical problem: If you break a stick at 2 points at random (chosen uniformly), what is the probability that the 3 resulting sticks form a triangle?

WLOG: Assume length is 1. Let $X, Y \sim \mathrm{Unif}([0,1])$



x, $y-x$, $1-y$ are the 3 sides of a triangle if and only if they satisfy the triangle inequality

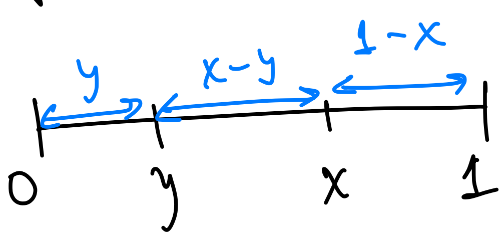
Recall: Triangle ineq.



$a < b + c$
$b < a + c$
$c < a + b$

$$\begin{cases} x < \cancel{y} - x + 1 - \cancel{y} \\ y - x < x + 1 - y \\ 1 - y < \cancel{x} + y - \cancel{x} \end{cases} \iff \begin{cases} x < 1/2 \\ y - x < 1/2 \\ y > 1/2 \end{cases}$$

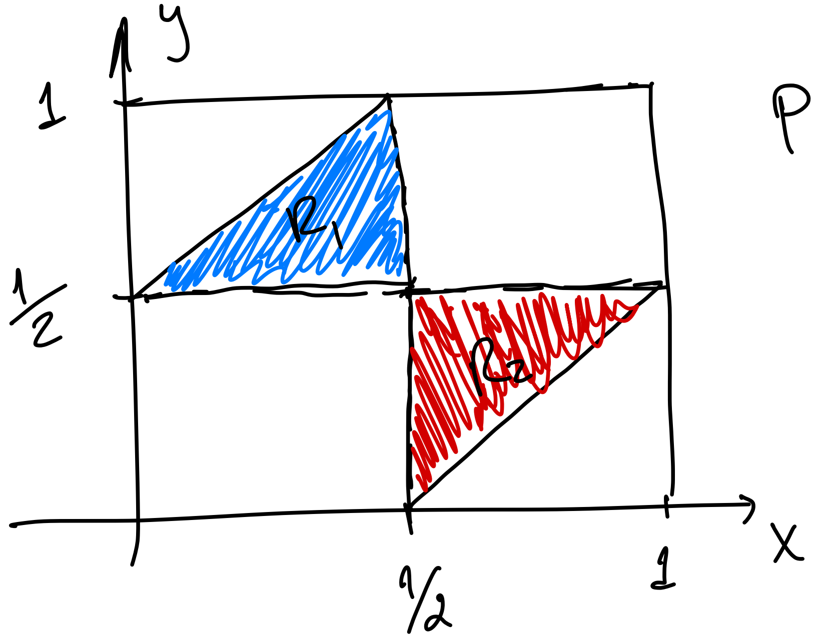
If the order is reversed:



$$\begin{cases} y < \cancel{x} - y + 1 - \cancel{x} \\ x - y < y + 1 - x \\ 1 - x < \cancel{y} + x - \cancel{y} \end{cases} \iff \begin{cases} y < 1/2 \\ x - y < 1/2 \\ x > 1/2 \end{cases}$$

3 resulting sticks form a triangle if and only if

$$\Big(\underbrace{x < \tfrac{1}{2} \text{ and } y - x < \tfrac{1}{2} \text{ and } y > \tfrac{1}{2}}_{R_1}\Big) \text{ or } \Big(\underbrace{y < \tfrac{1}{2} \text{ and } x - y < \tfrac{1}{2} \text{ and } x > \tfrac{1}{2}}_{R_2}\Big)$$



$$P\begin{pmatrix}\text{can form}\\ \text{a triangle}\end{pmatrix} = \iint_{R_1 \cup R_2} 1 \cdot dA$$

$$= \text{Area}(R_1 \cup R_2) = \frac{1}{4}$$

__Ans:__ 25%

__Independent Random Variables:__

__Recall:__ __Def:__ X, Y are independent if $\forall A, B$,

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B).$$

__Prop:__ Cont. rand. var. X, Y are independent if and only if their joint probability density function factors as

$$f_{X,Y}(x,y) = h(x)\, g(y), \quad \forall x, y \in \mathbb{R}.$$

__Pf:__ If X, Y are indep.,

$$P(X \le a, Y \le b) = P(X \le a)\, P(Y \le b)$$

$$\underbrace{\frac{\partial^2}{\partial a \partial b} P(X \le a, Y \le b)}_{f_{X,Y}(a,b)} = \underbrace{\frac{\partial^2}{\partial a \partial b} P(X \le a) P(Y \le b)}_{\frac{\partial}{\partial a}\left(\frac{\partial}{\partial b} P(X \le a) P(Y \le b)\right)}$$

$$f_{X,Y}(a,b) = \frac{\partial}{\partial a} P(X \leq a) \cdot \frac{\partial}{\partial b} P(Y \leq b)$$

$$= f_X(a) \cdot f_Y(b), \quad \forall a,b \in \mathbb{R}$$

Conversely, suppose

$$f_{X,Y}(x,y) = h(x)\, g(y) \quad \forall x,y \in \mathbb{R}$$

Then
$$P(X \in A, Y \in B) = \iint_{A \times B} f_{X,Y}(x,y)\, dA$$

$$= \int_A \int_B h(x) g(y) \; dy\, dx$$

$$= \int_A h(x) \left(\int_B g(y)\, dy \right) dx$$

$$= \int_A h(x)\, dx \cdot \int_B g(y)\, dy$$

$$\overset{\circledast}{=} P(X \in A) \cdot P(Y \in B).$$

$\square$

Claim:

$\circledast$ Here we are using that if $f_{X,Y}(x,y) = h(x) g(y)$, then, up to scaling $h(x) = f_X(x)$ and $g(y) = f_Y(y)$. This can be proven as follows:

If $f_{X,Y}(x,y) = h(x)\,g(y)$, we have

$$1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = \underbrace{\int_{-\infty}^{+\infty} h(x)\,dx}_{c_1} \cdot \underbrace{\int_{-\infty}^{+\infty} g(y)\,dy}_{c_2}$$

$$1 = c_1 \cdot c_2$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{+\infty} h(x)g(y)\,dy$$

$$= h(x) \cdot \underbrace{\int_{-\infty}^{+\infty} g(y)\,dy}_{c_2} = c_2 \cdot h(x)$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_{-\infty}^{+\infty} h(x)g(y)\,dx =$$

$$= g(y) \underbrace{\int_{-\infty}^{+\infty} h(x)\,dx}_{c_1} = c_1 \cdot g(y)$$

So

$$f_X(x) \cdot f_Y(y) = c_2\, h(x) \cdot c_1\, g(y) = h(x) \cdot g(x).$$

Therefore, up to scaling, h and g are the marginal p.d.f.'s of X and Y as claimed.

$\square$

Ex: Suppose the joint p.d.f. of X and Y is $f_{X,Y}(x,y)$. Are X and Y independent?

a) $f_{X,Y}(x,y) = 6e^{-2x}e^{-3y}$, $\quad 0<x<\infty,\ 0<y<\infty$

$$= \underbrace{6e^{-2x}}_{h(x)}\underbrace{e^{-3y}}_{g(y)}$$

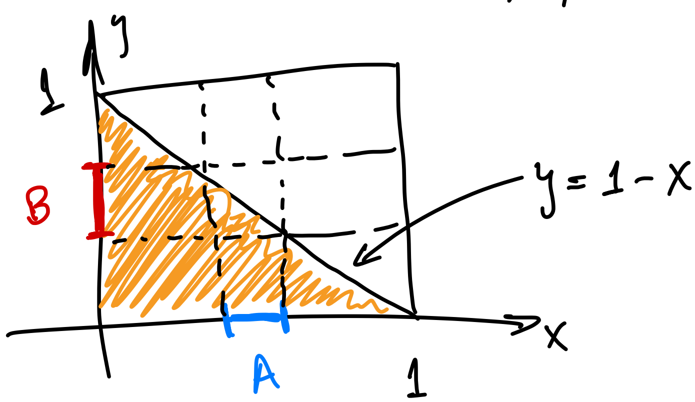
YES: X, Y are indep.

b) $f_{X,Y}(x,y) = 24xy$, $\quad 0<x<1,\ 0<y<1,\ 0<x+y<1$

$$= \underbrace{24x}_{h(x)}\underbrace{y}_{g(y)}$$

NO: X, Y are not indep.

Note: $P(X\in A, Y\in B) \neq P(X\in A)P(Y\in B)$ for some A, B



$$P(X\in A) = \int_A f_X(x)\,dx = \int_A\left[\int_0^1 f_{X,Y}(x,y)\,dy\right]dx$$

$$P(Y\in B) = \int_B f_Y(y)\,dy = \int_B\left[\int_0^1 f_{X,Y}(x,y)\,dx\right]dy$$

$$P(X\in A, Y\in B) = \iint_{A\times B} f_{X,Y}(x,y)\,dA$$

In other words:

$$f_{X,Y}(x,y) = 24xy \cdot I(x,y) \quad \forall x,y$$

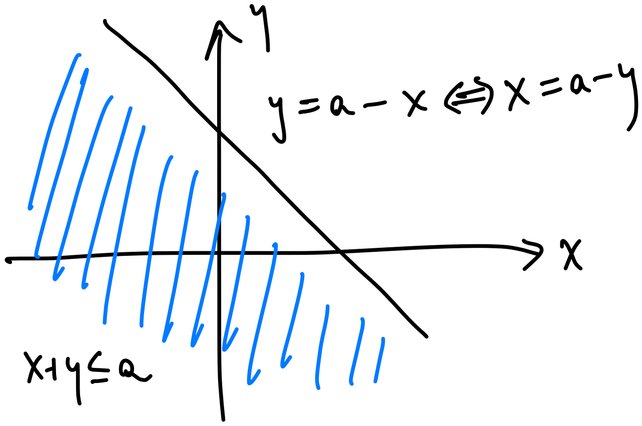
$$I(x,y) = \begin{cases} 1 & \text{if } 0<x<1,\ 0<y<1,\ 0<x+y<1 \\ 0 & \text{otherwise} \end{cases}$$

Sums of independent random variables:

X, Y indep. random variables $\left(f_{X,Y}(x,y) = f_X(x) f_Y(y) \right)$

Find the prob. density funct. f_{X+Y} of the sum $X+Y$.

$$F_{X+Y}(a) = P(X+Y \le a) = \iint_{x+y\le a} \underbrace{f_X(x) f_Y(y)}_{f_{X,Y}(x,y)} \, dA =$$



$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{a-y} f_X(x) f_Y(y) \, dx \, dy$$

$$= \int_{-\infty}^{+\infty} f_Y(y) \Big(\underbrace{\int_{-\infty}^{a-y} f_X(x)\, dx}_{F_X(a-y)} \Big) dy$$

$$= \int_{-\infty}^{+\infty} F_X(a-y) f_Y(y) \, dy$$

To get the p.d.f., differentiate:

$$f_{X+Y}(a) = \frac{d}{da} F_{X+Y}(a) = \frac{d}{da} \int_{-\infty}^{+\infty} F_X(a-y) f_Y(y)\, dy$$

$$= \int_{-\infty}^{+\infty} \frac{d}{da} F_X(a-y) f_Y(y) \, dy = \int_{-\infty}^{+\infty} f_X(a-y) f_Y(y)\, dy$$

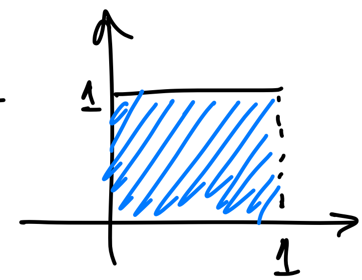
<u>Upshot</u>: $f_{X+Y}(a) = (f_X * f_Y)(a)$ "convolution of f_X and f_Y"

Example: Suppose $X, Y \sim \mathrm{Unif}([0,1])$ are independent.

Compute f_{X+Y} explicitly.

Note: $X+Y$ takes values in $[0,2]$.

$$f_X(a) = f_Y(a) = \begin{cases} 1 & \text{if } 0 \le a \le 1 \\ 0 & \text{otherwise} \end{cases}$$

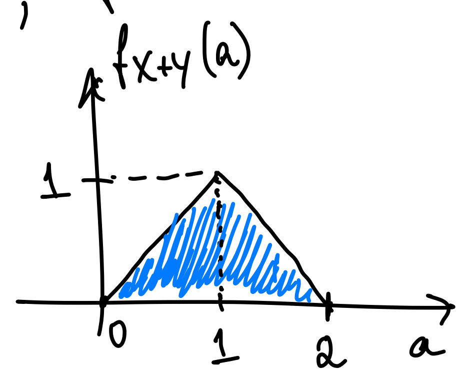


$$f_{X+Y}(a) = (f_X * f_Y)(a) = \int_{-\infty}^{+\infty} f_X(a-y) f_Y(y)\, dy$$

$$= \int_0^1 f_X(a-y) \underbrace{f_Y(y)}_{1}\, dy = \int_0^1 f_X(a-y)\, dy$$

$$= \begin{cases} \int_0^a 1\, dy = a, & \text{if } 0 \le a \le 1 \\ \int_{a-1}^1 1\, dy = 1-(a-1) = 2-a, & \text{if } 1 < a \le 2 \end{cases}$$

$$f_{X+Y}(a) = \begin{cases} a & \text{if } 0 \le a \le 1 \\ 2-a & \text{if } 1 < a \le 2 \\ 0 & \text{otherwise} \end{cases}$$



Careful: Sums of uniform random variables are **not** uniform!
(Convolution of constant functions is **not** constant)

But:
- Sums of **normal** random variables are normal:

$$X_1 \sim \text{Normal}(\mu_1, \sigma_1), \quad X_2 \sim \text{Normal}(\mu_2, \sigma_2)$$

$$\Rightarrow \quad X_1 + X_2 \sim \text{Normal}\left(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right)$$

- Sums of **Poisson** random variables are Poisson:

$$X_1 \sim \text{Poisson}(\lambda_1), \quad X_2 \sim \text{Poisson}(\lambda_2)$$

$$\Rightarrow \quad X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

- Sums of **exponential** random variables are **NOT** exponential. (they are actually "Gamma" distr.)