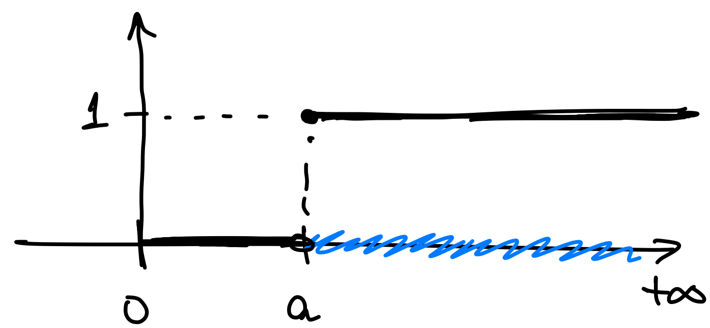
Towards the Central Limit Theorem ...

Markov's inequality: If X is a random variable that only assumes nonnegative values, then $\forall a > 0$

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Pf: Let $I : [0, \infty) \to \{0,1\}$ be the indicator function of $[a, +\infty)$, that is,

$$I(x) = \begin{cases} 1 & \text{if } x \geq a \\ 0 & \text{otherwise} \end{cases}$$



Note that $E(I(X)) = \int_0^{+\infty} I(x) f(x)\, dx = \int_a^{+\infty} f(x)\, dx$

$$= P(X \geq a)$$

Since $X \geq 0$, $I(X) \leq \frac{X}{a}$. Take expected values of both sides to get:

$$P(X \geq a) = E(I(X)) \leq E\left(\frac{X}{a}\right) = \frac{E(X)}{a}.$$

$\square$

Chebyshev's Inequality: If X is a random variable with finite mean μ and variance σ^2. Then $\forall k>0$

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

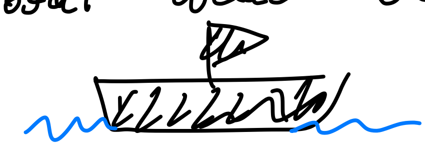
Pf: $|X-\mu| \geq k \iff |X-\mu|^2 \geq k^2$

Using Markov's ineq. with $a=k^2$ and X replaced by $|X-\mu|^2$, we get

$$P(|X-\mu| \geq k) = P(|X-\mu|^2 \geq k^2) \leq \frac{E(|X-\mu|^2)}{k^2} = \frac{\sigma^2}{k^2}.$$

$\square$

Ex: Suppose that a fishing boat collects 50 fish each week, on average.



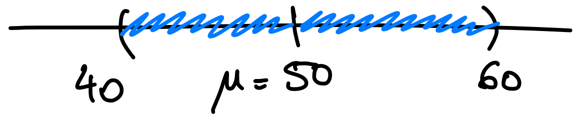
a) How large is the probability that the fishing boat collects more than 75 fish in a given week?

X = # fish collected in a week $\qquad E(X) = 50.$

$$P(X \geq 75) \overset{\text{Markov}}{\leq} \frac{E(X)}{75} = \frac{50}{75} = \frac{2}{3}.$$

b) If the variance in the number of fish collected each week is 25, find a lower bound for the probability that the number of fish collected in a week is between 40 and 60.

$\sigma^2 = 25$



$$P(40 \le X \le 60) = P(|X - 50| \le 10) = 1 - P(|X-50| \ge 10)$$

By Chebyshev: $P(|X-50| \ge 10) \le \dfrac{25}{10^2} = \dfrac{1}{4}$

Thus $P(40 \le X \le 60) \ge 1 - \frac{1}{4} = \frac{3}{4}$.

<u>Theorem</u> (Weak Law of Large Numbers). Let $X_1, X_2, X_3, \ldots$ be a sequence of independent identically distributed (iid) random variables with finite mean $\mu = E(X_i)$. Then $\forall \varepsilon > 0$

$$\lim_{n \to \infty} P\left(\left| \underbrace{\frac{X_1 + X_2 + \cdots + X_n}{n}}_{\bar{X}_n} - \mu \right| \ge \varepsilon \right) = 0$$

<u>Pf:</u> (Assume σ^2 is finite). Let $\bar{X}_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$.

Then $E(\bar{X}_n) = E\left(\dfrac{X_1 + \cdots + X_n}{n}\right) = \dfrac{E(X_1) + \cdots + E(X_n)}{n} = \dfrac{\cancel{n}\mu}{\cancel{n}} = \mu$

$$\mathrm{Var}(\overline{X}_n) = \mathrm{Var}\left(\frac{X_1 + \cdots + X_n}{n}\right) \underset{\substack{\uparrow \\ \text{indep.}}}{=} \frac{\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

By Chebyshev applied to $\overline{X}_n$, we have

$$0 \leq \lim_{n\to\infty} P\left(\left|\overline{X}_n - \mu\right| \geq \varepsilon\right) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0.$$

Thus $\lim_{n\to\infty} P\left(\left|\overline{X}_n - \mu\right| \geq \varepsilon\right) = 0.$ $\square$

(This means that $\overline{X}_n \xrightarrow{P} \mu$ as $n \to \infty$)

Strong Law of Large Numbers: Let $X_1, X_2, \ldots$ be iid random variables, with finite mean $\mu = E(X_i)$. Then

$$P\left(\lim_{n\to\infty} \overline{X}_n = \mu\right) = 1, \text{ where } \overline{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

(This means that $\overline{X}_n \xrightarrow{a.s.} \mu$ as $n \to \infty$)

Central Limit Theorem (Baby Version): Let $X_1, X_2, \ldots$ be a sequence of iid random variables with mean $\mu = E(X_i)$ and variance $\sigma^2 = Var(X_i)$. Then for all $a \in \mathbb{R}$,

$$\lim_{n \to \infty} P\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \leq a \right) = P(Z \leq a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-z^2/2} \, dz$$

where $\overline{X}_n = \dfrac{X_1 + \cdots + X_n}{n}$ and $Z \sim \text{Normal}(0,1)$.

Lemma: Let $Z_1, Z_2, \ldots$ be a sequence of random variables with distribution function F_{Z_n} and moment generating function M_{Z_n}. Let Z be a random variable with distribution function F_Z and moment generating function M_Z. If $M_{Z_n}(t) \to M_Z(t)$ for all t, then $F_{Z_n}(t) \longrightarrow F_Z(t)$ for all t at which $F_Z(t)$ is continuous.

Ex: If $Z \sim \text{Normal}(0,1)$, then $M_Z(t) = e^{t^2/2}$.

If Z_n is a sequence such that $M_{Z_n}(t) \to e^{t^2/2}$,

then $F_{Z_n}(t) \longrightarrow F_Z(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-z^2/2} \, dz$.

Pf of Central Limit Theorem: Assume $\mu = 0$ and $\sigma = 1$.

Suppose $M(t)$ is the moment generating function of each X_i. Note that $M(0) = 1$, $M'(0) = E(X_i) = \mu = 0$ and $M''(0) = E(X_i^2) = 1$. Moreover, the moment generating function of $\frac{X_i}{\sqrt{n}}$ is $E\left(e^{t\frac{X_i}{\sqrt{n}}}\right) = M\left(\frac{t}{\sqrt{n}}\right)$.

By independence, the moment generating function of

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \text{ is } M\left(\frac{t}{\sqrt{n}}\right) \cdot M\left(\frac{t}{\sqrt{n}}\right) \cdots M\left(\frac{t}{\sqrt{n}}\right) = \left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n .$$

Note
$$\frac{\overline{X}_n - \cancelto{0}{\mu}}{\cancelto{1}{\sigma}/\sqrt{n}} = \frac{\frac{X_1 + \cdots + X_n}{n} - 0}{1/\sqrt{n}} = \frac{X_1 + \cdots + X_n}{\sqrt{n}} .$$

By the Lemma, it suffices to show that $\left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n \to e^{t^2/2}$ for all t, as $n \to \infty$.

Let $L(t) = \log M(t)$, note that $L(0) = \log M(0) = 0$.

$$L'(0) = \frac{M'(0)}{M(0)} = \frac{\mu}{1} = \mu = 0$$

$$L''(0) = \frac{M(0)M''(0) - M'(0)^2}{M(0)^2} = \frac{1 . E(X^2) - \mu^2}{1} = E(X^2) = 1 .$$

Since $L(0) = L'(0) = 0$, using L'Hospital Rule, we compute:

$t n^{-1/2}$

$$\lim_{n\to\infty} \frac{L(t/\sqrt{n})}{n^{-1}} \overset{\text{L'H.}}{=} \lim_{n\to\infty} \frac{+L'(t/\sqrt{n})\cdot t\, n^{-3/2}\cdot n^{3/2}}{+n^{-2}\cdot 2\cdot n^{3/2}} =$$

$$= \lim_{n\to\infty} \frac{L'(t/\sqrt{n})\, t}{2n^{-1/2}} = \lim_{n\to\infty} \frac{+L''(t/\sqrt{n})\, t^2\cdot \cancel{n^{-3/2}}}{+2\cdot\cancel{\frac{2}{2}}\,\cancel{n^{-3/2}}}$$

$$= \lim_{n\to\infty} \frac{\underbrace{L''\left(\frac{t}{\sqrt{n}}\right)}_{\to 1} t^2}{2} = \frac{t^2}{2}.$$

Thus
$$\left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n = \left[e^{L(t/\sqrt{n})}\right]^n = e^{nL(t/\sqrt{n})} \longrightarrow e^{t^2/2},$$

$L(t) = \log M(t)$

$e^{L(t)} = M(t)$

$\forall t \in \mathbb{R}$, as we wanted to show!

To prove the result for general X_i, without assuming $\mu = 0$ and $\sigma = 1$, apply the above argument to

$$\tilde{X}_i = \frac{X_i - \mu}{\sigma},$$

noting that $E(\tilde{X}_i) = 0$ and $\mathrm{Var}(\tilde{X}_i) = 1$.

$\square$