MAT320/640, Fall 2021 **Renato Ghini Bettiol**

Homework Set 0 / Mock Assignment

DUE: AUG 30, 2021 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. If a is an even number, and b is an odd number, is $a+b$ even or odd? Explain.

2. Compute the 21^{st} derivative of the function $f(x) = x^{17} + 5x^4 - x^2 + 1$.

3. Explain (even if heuristically) how to compute the result of the following sum:

$$1 + 2 + 3 + \cdots + 98 + 99 + 100 \; = ?$$

Solutions

Text in blue represents side comments that are not integral parts of proofs, but address issues that some students might have had difficulties with in their attempted solutions.

1. First, recall the following:

   Definition 1. An integer $x \in \mathbb{Z}$ is called *even* if it can be written as $x = 2n$ for some $n \in \mathbb{Z}$, and *odd* if it can be written as $x = 2n+1$ for some $n \in \mathbb{Z}$.

   Now we are ready to solve the problem:

   Proof. If a is even and b is odd, then this means there exist $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ such that $a = 2n$ and $b = 2m+1$. (Note that we cannot use $b = 2n+1$ because the quantity n appearing there might be different from the one appearing in $a = 2n$.) Thus,

   $$a + b = 2n + 2m + 1 = 2(n+m) + 1,$$

   where $n + m \in \mathbb{Z}$, which means that $a+b$ is an odd number. □

   (Note that proof ended already, there is no need to analyze what happens if you divide both sides of the above equation by 2.)

2. In order to solve this problem, we first prove the following:

   Proposition. *For all $n \in \mathbb{N}$, if $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then the $(n+1)^{th}$ derivative of $p(x)$ vanishes identically: $p^{(n+1)}(x) = 0$.*

Proof (by induction). Let P_n be the statement that *if $p(x)$ is a polynomial of degree n, then the $(n+1)^{th}$ derivative of $p(x)$ vanishes identically: $p^{(n+1)}(x) = 0$.*

Basis: The statement P_1 holds because any polynomial of degree 1 is simply an affine function $p(x) = a_1 x + a_0$, whose first derivative is the constant function $p'(x) = a_1$, and whose second derivative is therefore identically zero, $p''(x) = 0$.

Step: Suppose P_n is true, and let $p(x) = a_{n+1}x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of degree $n+1$. Define $q(x) := p(x) - a_{n+1}x^{n+1}$ and note that $q(x)$ is a polynomial of degree n. Thus, by P_n, the $(n+1)^{th}$ derivative of $q(x)$ vanishes identically, i.e., $q^{(n+1)}(x) = 0$. The $(n+1)^{th}$ derivative of $p(x)$ can thus be computed:

$$p^{(n+1)}(x) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\big(a_{n+1}x^{n+1} + q(x)\big) = a_{n+1}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}x^{n+1} + q^{(n+1)}(x) = a_{n+1}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}x^{n+1},$$

where, in the second equality we used that the derivative is linear,[1] and in the third equality we used that $q^{(n+1)}(x) = 0$. As you learnt in Calculus,[2]

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}x^{n+1} = (n+1)!,$$

which, together with the above, implies that $p^{(n+1)}(x) = a_{n+1}(n+1)!$ is a constant function. Therefore, its derivative, which is the $(n+2)^{th}$ derivative of $p(x)$, vanishes identically: $p^{(n+2)}(x) = 0$. This proves that P_{n+1} holds, and concludes the proof. □

As an application of the above Proposition (using $n = 17$), the 18^{th} derivative of $f(x) = x^{17} + 5x^4 - x^2 + 1$ vanishes identically, since $f(x)$ is a polynomial of degree 17. Thus, also the 21^{st} derivative vanishes identically.

(Note that no actual computation with ugly numbers is required here. Instead, we proved a much more general statement, and then it can simply be quoted as needed. Note also that any statements where you informally use "..." is, behind the scenes, a proof by induction like the one above. In this class, you will be expected to *at least* be sensitive to such subtleties and acknowledge there is something – like a proof by induction – being implicitly used in the background, even if you do not provide such a detailed proof, like I did above.)

3. We proved in Lecture 1 (by induction) that, for all $n \in \mathbb{N}$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

and you should revise that proof if needed. Applying that result[3] with $n = 100$, it follows that $1 + 2 + \cdots + 99 + 100 = \frac{10100}{2} = 5050$.

[1] This means that $(af + bg)' = af' + bg'$ if a, b are constants and f, g are functions.

[2] The fact that the n^{th} derivative of $f(x) = x^n$ is $f^{(n)}(x) = n!$ can also be (separately and independently) proven by induction on $n \in \mathbb{N}$, and would be a great follow-up exercise.

[3] Whenever something was proven in class, you may use it without having to repeat the proof again.