MAT320/640, Fall 2021 **Renato Ghini Bettiol**

Homework Set 3

DUE: OCT 13, 2021 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. Write a detailed and rigorous proof (i.e., finding $N \in \mathbb{N}$ in terms of the given $\varepsilon > 0$) that the sequence
$$s_n = 2021\left(1 + \frac{(-1)^n}{n}\right), \quad n \in \mathbb{N}$$
converges to $L = 2021$.

2. Regarding the above sequence $s_n = 2021\left(1 + \frac{(-1)^n}{n}\right)$, answer (with justification) each of the following questions:
   (a) Is $(s_n)_{n\in\mathbb{N}}$ bounded?
   (b) Is $(s_n)_{n\in\mathbb{N}}$ a Cauchy sequence?
   (c) Does $(s_n)_{n\in\mathbb{N}}$ have a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ that converges to 0?

Solutions

Text in blue represents side comments that are not integral parts of proofs, but address issues that some students might have had difficulties with in their attempted solutions.

1. *Proof.* In order to prove that $s_n \to L$, we must show that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|s_n - L| < \varepsilon$. We begin by computing:
$$|s_n - L| = \left|\underbrace{2021\left(1 + \frac{(-1)^n}{n}\right)}_{s_n} - \underbrace{2021}_{L}\right| = 2021\left|1 - \frac{(-1)^n}{n} - 1\right| = \frac{2021}{n}.$$
Thus, in order to ensure $|s_n - L| < \varepsilon$, it suffices to request that $\frac{2021}{n} < \varepsilon$, equivalently, $n > \frac{2021}{\varepsilon}$. So we let N be the smallest integer larger than $\frac{2021}{\varepsilon}$. With this choice of N for each given $\varepsilon > 0$, we have that $|s_n - L| < \varepsilon$ whenever $n \geq N$, as requested. □

2. [Many students attempted to reprove some of the results below; please remember you can directly use the results we proved in lecture, if you quote them correctly.]

   (a) **Yes**, the sequence $(s_n)_{n\in\mathbb{N}}$ is bounded, because it is convergent.
   [We proved in Lecture 6 that every convergent sequence is bounded.]

(b) **Yes**, the sequence $(s_n)_{n\in\mathbb{N}}$ is Cauchy, because it is convergent.
[We proved in Lecture 8 that a sequence of real numbers is Cauchy if and only if it is convergent.]

(c) **No**, the sequence $(s_n)_{n\in\mathbb{N}}$ does not have any subsequence that converges to 0, because every subsequence of $(s_n)_{n\in\mathbb{N}}$ converges to the same limit $L = 2021$ as $(s_n)_{n\in\mathbb{N}}$.